ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Advances in Mathematics 228 (2011) 2403–2440

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Asymptotic properties of random matrices and pseudomatrices ☆

Romuald Lenczewski

Instytut Matematyki i Informatyki, Politechnika Wrocławska, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

Received 6 January 2010; accepted 19 July 2011

Available online 7 August 2011

Communicated by Dan Voiculescu

Abstract

We study the asymptotics of sums of matricially free random variables, called random pseudomatrices, and we compare it with that of random matrices with block-identical variances. For objects of both types we find the limit joint distributions of blocks and give their Hilbert space realizations, using operators called 'matricially free Gaussian operators'. In particular, if the variance matrices are symmetric, the asymptotics of symmetric blocks of random pseudomatrices agrees with that of symmetric random blocks. We also show that blocks of random pseudomatrices are 'asymptotically matricially free' whereas the corresponding symmetric random blocks are 'asymptotically symmetrically matricially free', where symmetric matricial freeness is obtained from matricial freeness by an operation of symmetrization. Finally, we show that row blocks of square, block-lower-triangular and block-diagonal pseudomatrices are asymptotically free, monotone independent and boolean independent, respectively.

MSC: 46L53; 46L54; 15B52

Keywords: Freeness; Matricial freeness; Symmetric matricial freeness; Matricially free Gaussian operator; Random matrix; Random pseudomatrix

☆ This work is supported by the Polish Ministry of Science and Higher Education research grant N N201 364436.

E-mail address: romuald.lenczewski@pwr.wroc.pl.

0001-8708/$ – see front matter

doi:10.1016/j.aim.2011.07.011

1. Introduction and main results

We have recently shown that the Hilbert space construction of the free product of states on C^*-algebras given by Voiculescu [8] can be generalized to a framework which exhibits some matricial features [5].

We considered arrays of noncommutative probability spaces, for which we defined the *matricially free product of states*. The definition of this product is based on replacing a family of canonical unital $*$-representations of C^*-algebras on the free product of Hilbert spaces by a diagonal-containing array of non-unital $*$-representations of C^*-algebras on the matricially free product of Hilbert spaces. The crucial point is that products of these representations imitate products of matrices rather than free products, although the main features of the latter are still present. We studied the associated concepts of noncommutative independence called *matricial freeness* and related to it *strong matricial freeness* which can be viewed as a scalar-type generalization of freeness underlying other fundamental notions of noncommutative independence (monotone and boolean) and some of their generalizations (conditional freeness and conditional monotone independence).

More importantly, matricial freeness is closely related to random matrices. Their significance for free probability was discovered by Voiculescu [9], who showed that Gaussian random matrices with mutually independent entries are asymptotically free. This connection was later generalized by Dykema [2] to non-Gaussian random matrices. Of course, the first indication that free probability might be related to random matrices was the central limit theorem for free random variables since its limit law was the semicircle law obtained in the classical work of Wigner [13] as the limit distribution of certain self-adjoint random matrices with independent entries.

The main objects of interest in our study of matricial freeness and strong matricial freeness are diagonal-containing arrays of (in general, non-unital) subalgebras of a given unital algebra $\mathcal{A}$,

$$(\mathcal{A}_{i,j})_{(i,j)\in J} \quad \text{with } \Delta \subseteq J \subseteq I \times I, \text{ where } \Delta = \big\{(j,j)\colon\ j \in I\big\},$$

equipped with an array of states $(\varphi_{i,j})$ on $\mathcal{A}$, with respect to which notions of noncommutative independence are defined. Here, the states $\varphi_{i,j}$ and their kernels play a very similar role to that of one distinguished state φ and its kernel in free probability. Apart from square arrays, of particular interest are lower- (upper-) triangular arrays related to monotone (anti-monotone) independence introduced by Muraki [6]. In all arrays, the roles of diagonal and off-diagonal entries are quite different, which is a characteristic feature of this theory reminding the random matrix theory.

Let $(X_{i,j}(n))$ be an n-dimensional square array (or its diagonal-containing subarray) of self-adjoint random variables in a unital $*$-algebra $\mathcal{A}(n)$ which is matricially free with respect to the array $(\phi_{i,j}(n))$ defined in terms of a family $(\phi_j(n))_{1\leqslant j\leqslant n}$. The sums

$$S(n) = \sum_{(i,j)\in J(n)} X_{i,j}(n),$$

where $(J(n))$ is an appropriate sequence of sets, called *random pseudomatrices*, remind random matrices, whereas convex linear combinations

$$\psi(n) = \frac{1}{n}\sum_{j=1}^{n} \phi_j(n),$$

where $n \in \mathbb{N}$, replace the states $\tau(n) = \mathbb{E} \otimes \mathrm{tr}(n)$, classical expectation tensored with normalized traces, under which distributions of random matrices are computed.

Under suitable assumptions on the distributions of the variables $X_{i,j}(n)$ in the states $\phi_{i,j}(n)$, respectively, we can now study the asymptotic distributions of sums corresponding to blocks of random pseudomatrices in various states. These blocks are defined as sums of the form

$$S_{p,q}(n) = \sum_{(i,j) \in N_p \times N_q} X_{i,j}(n),$$

where $[n] := \{1, 2, \ldots, n\} = N_1 \cup N_2 \cup \cdots \cup N_r$ is a partition into disjoint subsets whose sizes grow proportionately to n (it is convenient to think of intervals). The dependence of the N_j's on n is suppressed in the notation. In particular, we assume that the variances $v_{i,j}(n) := \phi_{i,j}(n)(X_{i,j}^2(n))$ are identical within blocks.

This leads to various asymptotic properties of sums of matricially free and strongly matricially free random variables. Let $\phi_{i,j}(n)$ be defined according to

$$\phi_{j,j}(n) = \phi(n) \quad \text{and} \quad \phi_{i,j}(n) = \phi_j(n) \quad \text{for } i \neq j,$$

where $\phi(n)$ is a distinguished state on $\mathcal{A}(n)$ and $(\phi_j(n))_{1 \leqslant j \leqslant n}$ is a family of additional states called *conjugate states* associated with $\phi(n)$ for any $n \in \mathbb{N}$ (they were called 'conditions' in [5]). Then, as $n \to \infty$, we obtain the following properties:

(i) if the variances of square arrays are identical within blocks and rows, then the sums

$$A_p(n) := \sum_{q=1}^{n} S_{p,q}(n), \quad \text{where } 1 \leqslant p \leqslant r,$$

are asymptotically free with respect to $\phi(n)$,

(ii) if the variances of block-lower-triangular arrays are identical within blocks and rows, then the sums

$$B_p(n) := \sum_{q=1}^{p} S_{p,q}(n), \quad \text{where } 1 \leqslant p \leqslant r,$$

are asymptotically monotone independent with respect to $\phi(n)$,

(iii) if the variances of block-diagonal arrays are identical within blocks, then the sums $C_p(n) := S_{p,p}(n)$, where $1 \leqslant p \leqslant r$, are asymptotically boolean independent with respect to $\phi(n)$.

Let us add that non-asymptotic analogs of (i)–(iii) for finite sums of *strongly* matricially free random variables were proved in [5] and that (iii) also holds for finite sums of matricially free random variables. Recall that the 'strongly matricially free product of states', on which the definition of strong matricial freeness is based, is obtained from the matricially free product by restriction. However, it is worth to remark that the difference between blocks of matricially free and strongly matricially free random variables disappears asymptotically.

The scheme of matricial freeness and its symmetrized version, in which ordered pairs are replaced by (non-ordered) sets consisting of one or two elements, called *symmetric matricial* *freeness*, is also used for arrays $(\phi_{i,j}(n))$ defined in terms of a family $(\phi_j(n))_{1\leqslant j\leqslant n}$ of states on $\mathcal{A}(n)$ according to

$$\phi_{i,j}(n)=\phi_j(n) \quad \text{for any } i, j,$$

and any $n\in\mathbb{N}$. Here, we do not a priori assume that the states $\phi_j(n)$ are conjugate states associated with a distinguished state. In the study of asymptotics, of importance become arrays defined by 'normalized partial traces'

$$\psi_q(n)=\frac{1}{n_q}\sum_{j\in N_q}\phi_j(n)$$

according to $\psi_{p,q}(n)=\psi_q(n)$ for any $p,q\in[r]$ and $n\in\mathbb{N}$, where n_q is the cardinality of N_q. Then, as $n\to\infty$, we obtain the following properties:

(iv) if the variances are identical within blocks, then the array $(S_{p,q}(n))$ is asymptotically matricially free with respect to $(\psi_{p,q}(n))$,
(v) if the variances are identical within blocks and form symmetric matrices, then the array $(Z_{p,q}(n))$ of symmetric blocks given by

$$Z_{p,q}(n):=\sum_{(i,j)\in N_{p,q}}X_{i,j}(n)$$

where $N_{p,q}=(N_p\times N_q)\cup(N_q\times N_p)$, is asymptotically symmetrically matricially free with respect to $(\psi_{p,q}(n))$,
(vi) if the variances are identical within blocks of symmetric random matrices, then the array $(T_{p,q}(n))$ of symmetric random blocks is asymptotically symmetrically matricially free with respect to $(\tau_{p,q}(n))$,

where the array $(\tau_{p,q}(n))$ is defined by the family of partial traces $\tau_q=\mathbb{E}\otimes\mathrm{tr}_q(n)$ and $\mathrm{tr}_q(n)$ denotes the normalized trace over the vectors indexed by N_q. Here, by symmetric random blocks we understand symmetric blocks of symmetric random matrices in the approach of Voiculescu [9] and Dykema [2].

Concerning a non-asymptotic analog of (iv), we show in this paper that blocks $(S_{p,q}(n))$ are matricially free for any finite n under the stronger assumption that the variables have block-identical distributions. In turn, properties (v) and (vi) do not seem to have their non-asymptotic analogs. Nevertheless, they are the reasons why we call the sums $S(n)$ random pseudomatrices. On the other hand, let us also remark that we do not have an analog of (iv) for random matrices. Therefore, informally, random pseudomatrices can be viewed as objects which remind random matrices for large n if we consider blocks with symmetric variances, but exhibit different features if the variances are not symmetric, which leads to triangular arrays and relations to monotone independence.

In that connection we would like to mention the result of Shlyakhtenko [7] who established a connection between a class of symmetric random matrices called random band matrices and an operator-type generalization of freeness called freeness with amalgamation. Our result (vi) gives a connection between a slightly smaller class of symmetric random matrices called symmetric random blocks with block-identical distributions and a scalar-type generalization of freeness called symmetric matricial freeness. Although we have not yet established a relation between both approaches, one can expect that some relation between these two notions of independence can be found.

In our previous work, we expressed the limit distributions of random pseudomatrices with respect to $\phi(n)$'s and $\psi(n)$'s in terms of certain functions on the class of colored non-crossing pair partitions, as well as in terms of some 'continued multifractions' [5]. Both these realizations showed that the limit laws can be viewed as matricial generalizations of the semicircle laws. Moreover, these distributions turned out to be related to s-free additive convolutions and the associated notion of freeness with subordination, or s-freeness [3], concepts motivated by the results of Voiculescu [10] and Biane [1] on analytic subordination in free probability. Using s-freeness, we also studied s-free and free multiplicative convolutions, which allowed us to establish relations between these convolutions (as well as some other multiplicative convolutions) and certain classes of walks on appropriately defined products of graphs [4].

In this paper, we give Hilbert-space realizations of the limit distributions of random pseudomatrices and their blocks under $\phi(n)$, $\psi_k(n)$ and $\psi(n)$, where the underlying Hilbert space in all considered cases is the matricially free product

$$(\mathcal{F},\xi)=*_{i,j}^{M}(\mathcal{F}_{p,q})$$

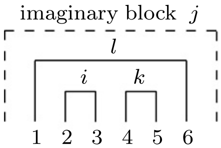
of the r-dimensional array $(\mathcal{F}_{p,q})$ of Fock spaces, where the diagonal and off-diagonal Fock spaces are, respectively, free and boolean Fock spaces over one-dimensional Hilbert spaces. Using realizations on $\mathcal{F}$ and operators which play the role of Gaussian operators in our approach, as well as their 'symmetrizations', we prove (i)–(vi).

In Section 2, we present the combinatorics of colored non-crossing pair-partitions. In Section 3, we recall the basic notions and facts concerning matricially free random variables. In Section 4, we introduce matricially free Gaussian operators on $\mathcal{F}$. Using them, we find $\mathcal{F}$-realizations of the limit distributions for random pseudomatrices in Section 5. In Section 6, we prove that blocks of matricially free arrays of random variables with block-identical distributions are matricially free with respect to a suitably defined array of states. In turn, asymptotic matricial freeness of blocks of random pseudomatrices is proved in Section 7, where we also find $\mathcal{F}$-realizations of their limit joint distributions. In Section 8, we introduce the notion of 'symmetric matricial freeness'. In Section 9, we find the limit joint distributions of symmetric random blocks and their $\mathcal{F}$-realizations and we show that symmetric random blocks are asymptotically symmetrically matricially free. In Section 10, we obtain results on asymptotic freeness and asymptotic monotone independence of rows of pseudomatrices.

2. Combinatorics

The combinatorics of our model is based on the class of *colored non-crossing pair partitions*, to which we assign certain products of matrix elements whose indices depend on the colorings of their blocks.

For a given non-crossing pair partition π, we denote by $\mathcal{B}(\pi)$, $\mathcal{L}(\pi)$ and $\mathcal{R}(\pi)$ the sets of its blocks, their left and right legs, respectively. If $\pi_i=\{l(i),r(i)\}$ and $\pi_j=\{l(j),r(j)\}$ are blocks of π with left legs $l(i)$ and $l(j)$ and right legs $r(i)$ and $r(j)$, respectively, then π_i is *inner* with respect to π_j if $l(j)<l(i)<r(i)<r(j)$. In that case π_j is *outer* with respect to π_i. It is the *nearest outer block* of π_i if there is no block $\pi_k=\{l(k),r(k)\}$ such that $l(j)<l(k)<l(i)<r(i)<r(k)<r(j)$. Since the nearest outer block, if it exists, is unique, we can write in this case



blocks $\pi_1=\{1,6\},\pi_2=\{2,3\},\pi_3=\{4,5\}$
legs $\mathcal{L}(\pi)=\{1,2,4\}$, $\mathcal{R}(\pi)=\{3,5,6\}$
coloring $f(\pi_1)=l$, $f(\pi_2)=i$, $f(\pi_3)=k$
nearest outer blocks $o(\pi_2)=\pi_1$, $o(\pi_3)=\pi_1$

Fig. 1. A colored non-crossing partition.

$\pi_j = o(\pi_i)$, $l(j) = o(l(i))$ and $r(j) = o(r(i))$. We will denote by $I(\pi_j)$ the union of π_j and all blocks for which π_j is the nearest outer block. If π_0 is the imaginary block, we will understand that $I(\pi_0)$ is the union of π_0 and all covering blocks. If $\pi_j = o(\pi_i)$ and $\pi_k = o(\pi_j)$, the triple (π_i, π_j, π_k) will be called the *inner–outer triple*. If π_i does not have an outer block, it is called a *covering* block. It is convenient to extend each partition $\pi \in \mathcal{NC}_m^2$ to the partition $\hat{\pi}$ obtained from π by adding one block, say $\pi_0 = \{0, m+1\}$, called the *imaginary block*.

Let $F_r(\pi)$ be the set of all mappings $f : \mathcal{B}(\pi) \to [r]$ called *colorings* of the blocks of π by the set $[r] := \{1, 2, \ldots, r\}$. Then the pair (π, f) plays the role of a colored partition. Its blocks will be denoted

$$\mathcal{B}(\pi, f) = \big\{(\pi_1, f), (\pi_2, f), \ldots, (\pi_k, f)\big\},$$

where we use pairs (π_i, f) since to each block π_i we shall assign entries of a matrix which depend on the colors of both π_i and $o(\pi_i)$ for any $i \in [k]$. If the imaginary block is used, it is convenient to assume that it is also colored by a number from the set $[r]$. In Fig. 1 we give an example with the notions defined above.

Definition 2.1. Let (π, f) be a colored non-crossing partition with blocks as above, where $f \in F_r(\pi)$ and let $B \in M_r(\mathbb{R})$ be given. For any $0 \leqslant j \leqslant r$, we define

$$b_j(\pi, f) = b_j(\pi_1, f)b_j(\pi_2, f)\ldots b_j(\pi_k, f),$$

where the functions $b_j : \mathcal{B}(\pi, f) \to \mathbb{R}$ are given by the following rules:

1. $b_j(\pi_i, f) = b_{p,q}$ if $f(\pi_i) = p$ and $f(o(\pi_i)) = q$, where $0 \leqslant j \leqslant r$,
2. $b_j(\pi_i, f) = b_{p,j}$ if $f(\pi_i) = p$ and π_i does not have outer blocks, where $1 \leqslant j \leqslant r$,
3. $b_0(\pi_i, f) = b_{p,p}$ if $f(\pi_i) = p$ and π_i does not have outer blocks.

The index $j \in [r]$ in $b_j(\pi, f)$ can be interpreted as the color of the imaginary block (if $j = 0$, then the imaginary block is not needed). Finally, $\mathcal{NC}_m^2 = \emptyset$ for m odd and thus we shall understand in this case that the summation over $\pi \in \mathcal{NC}_m^2$ gives zero.

When we sum these products over all possible colorings, we obtain numbers

$$b_j(\pi) = \sum_{f \in F_r(\pi)} b_j(\pi, f)$$

for any $0 \leqslant j \leqslant r$, which are used to express the limit laws of random pseudomatrices.

Example 2.1. If we consider the partition π given by the diagram in Fig. 1, we obtain

$$b_0(\pi) = \sum_{i,k,l} b_{i,l} b_{k,l} b_{l,l} \quad \text{and} \quad b_j(\pi) = \sum_{i,k,l} b_{i,l} b_{k,l} b_{l,j},$$

where $j \in [r]$ and all indices run over $[r]$, which corresponds to all possible colorings from $F_r(\pi)$ and is adequate for a square array $B \in M_r(\mathbb{R})$. For other arrays, we obtain the same formulas except that the summation range is smaller.

Entries of the matrix B will be related to variances of matricially free random variables obtained when computing their mixed moments. Therefore, it is natural to begin with an array $(a_{i,j})$ of random variables and assign a variable to each leg of the considered partition. If the array is square, we assume that $i, j \in [r]$, whereas in other cases the pairs (i, j) belong to a proper subset $J \subset [r] \times [r]$ which includes the diagonal. To a given non-crossing partition $\pi \in \mathcal{NC}_m^2$, where m is even, we can now associate a product of these variables in which indices can be interpreted as colors taken from the set $[r]$. These indices are related to each other in a natural way which refers to the way matricially free random variables can be multiplied to give a non-trivial contribution to the limit laws. The definition given below is based on this relation.

Definition 2.2. We will say that the partition $\pi \in \mathcal{NC}_m^2$ is *adapted* to a tuple of numbers $(p_1, q_1, \ldots, p_m, q_m)$ from the set $[r]$ if

(a) $(p_i, q_i) = (p_j, q_j)$ whenever $\{i, j\}$ is a block of π,
(b) $q_j = p_{o(j)}$ whenever $\{i, j\}$ is a block of π which has an outer block.

We denote by $\mathcal{NC}_m^2(p_1, q_1, \ldots, p_m, q_m)$ the set of partitions adapted to $(p_1, q_1, \ldots, p_m, q_m)$. In turn, by $\mathcal{NC}_{m,q}^2(p_1, q_1, \ldots, p_m, q_m)$ we will denote the subset of $\mathcal{NC}_m^2$ for which (a)–(b) hold and $q_j = q$ ($q_j = p_j$, respectively) whenever j belongs to a covering block, where $q \in [r]$ ($q = 0$).

If $\pi \in \mathcal{NC}_{m,q}^2(p_1, q_1, \ldots, p_m, q_m)$, then the tuple $(p_1, q_1, \ldots, p_m, q_m)$ defines a unique coloring of π, in which the block containing k is colored by p_k for any k (if $q \in [r]$, the imaginary block is colored by q).

Example 2.2. Consider the partition π shown in Fig. 1 and let $p_1, q_1, \ldots, p_6, q_6, q \in [6]$ be given. Conditions (a)–(b) of Definition 2.2 lead to equations $p_1 = p_6$, $p_2 = p_3$, $p_4 = p_5$ and $q_2 = q_3 = p_1$, $q_4 = q_5 = p_1$, $q_1 = q_6 = q$, which gives four independent colors. Setting $p_1 = l$, $p_2 = i$, $p_4 = k$ and $q = j$, we obtain the coloring of Fig. 1. Note in this context that p_1, p_2, p_4 may be interpreted as colors of the left legs of π and q_6 as the color of the imaginary block. These equations lead to a product of variables of the form

$$a_{p_1,q_6} a_{p_2,p_1} a_{p_2,p_1} a_{p_4,p_1} a_{p_4,p_1} a_{p_1,q_6},$$

where each variable is taken from an array $(a_{p,q})$ of matricially free random variables and is associated with one inner–outer pair of blocks of $\hat{\pi}$.

We close this section with the symmetrized version of Definition 2.2 which will be needed in a combinatorial formula for the limit joint distribution of symmetric random blocks. The symmetrization is obtained by replacing ordered pairs by subsets. Note that condition (b) of Definition 2.2 is replaced by two conditions (b)–(c) which are necessary conditions for products of random variables which belong to blocks of a matrix to be non-trivial.

Definition 2.3. We will say that the partition $\pi \in \mathcal{NC}_m^2$ is *adapted* to a tuple of subsets $(\{p_1, q_1\}, \dots, \{p_m, q_m\})$ of the set $[r]$ if

(a) $\{p_i, q_i\} = \{p_j, q_j\}$ whenever $\{i, j\}$ is a block of π,
(b) $\bigcap_{i \in I(\pi_k)} \{p_i, q_i\} \neq \emptyset$ whenever π_k is a block of π,
(c) $\{p_j, q_j\} \subseteq \{p_i, q_i\} \cup \{p_k, q_k\}$ whenever the blocks containing i, j, k, respectively, form an inner–outer triple.

The set of such partitions will be denoted $\mathcal{NC}_m^2(\{p_1, q_1\}, \dots, \{p_m, q_m\})$. The subset of $\mathcal{NC}_m^2$ for which (a)–(c) hold for all blocks of $\hat{\pi}$ with $\{q, q\}$ formally assigned to both legs of the imaginary block, where $q \in [r]$, will be denoted $\mathcal{NC}_{m,q}^2(\{p_1, q_1\}, \dots, \{p_m, q_m\})$.

If $\pi \in \mathcal{NC}_{m,q}^2(\{p_1, q_1\}, \dots, \{p_m, q_m\})$, the tuple $(\{p_1, q_1\}, \dots, \{p_m, q_m\})$ and the number q define colorings of π called *admissible*, in which each block containing k is colored by p_k or q_k and each covering block is colored by q.

Example 2.3. Consider again the partition shown in Fig. 1 and let $p_1, q_1, \dots, p_6, q_6 \in [6]$ be given. Definition 2.3 leads to equations

1. $\{p_1, q_1\} = \{p_6, q_6\}$, $\{p_2, q_2\} = \{p_3, q_3\}$ and $\{p_4, q_4\} = \{p_5, q_5\}$,
2. $\{p_1, q_1\} \cap \{p_3, q_3\} \cap \{p_5, q_5\} \neq \emptyset$.

For instance, they are satisfied if $(p_1, q_1) = (q_6, p_6)$, $(p_2, q_2) = (q_3, p_3)$, $(p_4, q_4) = (q_5, p_5)$, with $q_1 = q_3 = q_5$, which corresponds to the matricial product of random variables of the form

$$a_{p_1,p_6} a_{p_6,p_3} a_{p_3,p_6} a_{p_6,p_5} a_{p_5,p_6} a_{p_6,p_1},$$

where we have four independent indices p_1, p_3, p_5, p_6, among which p_1 colors the imaginary block and p_3, p_5, p_6 color the blocks of π (they are associated with the right legs of π). If we require that $\pi \in \mathcal{NC}_{6,q}^2(\{p_1, q_1\}, \dots, \{p_6, q_6\})$ and that the number of independent indices stays the same, it is necessary that $p_1 = q$. Let us add that we can replace some indices from the set $\{p_1, p_3, p_5, p_6\}$ by the corresponding q_i's in order to obtain other admissible colorings and the associated products of variables.

Example 2.4. Consider the partition $\sigma = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$ of the set $[6]$. In this case, conditions (a)–(c) of Definition 2.3 are satisfied if $(p_1, q_1) = (q_6, p_6)$, $(p_2, q_2) = (q_5, p_5)$, $(p_3, q_3) = (q_4, p_4)$ with $q_1 = q_5$ and $q_4 = p_5$, which corresponds to the matricial products of the form

$$a_{p_1,p_6} a_{p_6,p_5} a_{p_5,p_4} a_{p_4,p_5} a_{p_5,p_6} a_{p_6,p_1},$$

where again we have four independent indices p_1, p_4, p_5, p_6 (the last three are associated with the right legs of σ). In order that $\pi \in \mathcal{NC}^2_{6,q}(\{p_1, q_1\}, \ldots, \{p_6, q_6\})$, with the same number of independent indices, it is necessary that $p_1 = q$. Other admissible colorings are obtained by replacing some indices from the set of independent p_i's by the corresponding q_i's.

3. Matricially free random variables

Let us recall the definition of matricially free random variables as well as the main results of [5], where we refer the reader for details.

Let $\mathcal{A}$ be a unital algebra with an array $(\mathcal{A}_{i,j})$ of not necessarily unital subalgebras of $\mathcal{A}$ and let $(\varphi_{i,j})$ be a family of states on $\mathcal{A}$. Here, by a state on $\mathcal{A}$ we understand a normalized linear functional. Further, we assume that each $\mathcal{A}_{i,j}$ has an *internal unit* $1_{i,j}$, for which it holds that $a1_{i,j} = 1_{i,j}a = a$ for any $a \in \mathcal{A}_{i,j}$, and that the unital subalgebra $\mathcal{I}$ of $\mathcal{A}$ generated by all internal units is commutative. If $\mathcal{A}$ is a unital $*$-algebra, then, in addition, we require that the considered functionals are positive, the subalgebras are $*$-subalgebras and the internal units are projections.

However, the definitions given below are slightly more general than those in [5]. Namely, in contrast to the formulation given there, we do not assume any particular form of the considered array $(\varphi_{i,j})$. Instead, we will distinguish the *diagonal* states as those of the form $\varphi_{j,j}$, where $j \in I$, but we will not assume that they all coincide. This will enable us to use the concept of matricial freeness for a wider class of arrays, which turns out convenient in the formulation of our results.

We shall use the subsets of $(I \times I)^m$ of the form

$$\Lambda_m = \big\{\big((i_1, i_2), (i_2, i_3), \ldots, (i_m, i_{m+1})\big)\colon (i_1, i_2) \neq (i_2, i_3) \neq \cdots \neq (i_m, i_{m+1})\big\},$$

where $m \in \mathbb{N}$, with their union denoted $\Lambda = \bigcup_{m=1}^{\infty} \Lambda_m$.

Definition 3.1. We say that $(1_{i,j})$ is a *matricially free array of units* associated with $(\mathcal{A}_{i,j})$ and $(\varphi_{i,j})$ if for any diagonal state φ it holds that

1. $\varphi(u_1 a u_2) = \varphi(u_1)\varphi(a)\varphi(u_2)$ for any $a \in \mathcal{A}$ and $u_1, u_2 \in \mathcal{I}$,
2. if $a_k \in \mathcal{A}_{i_k,j_k} \cap \operatorname{Ker}\varphi_{i_k,j_k}$, where $1 < k \leqslant m$, then

$$\varphi(a1_{i_1,j_1}a_2 \ldots a_m) = \begin{cases} \varphi(aa_2 \ldots a_n) & \text{if } ((i_1, j_1), \ldots, (i_m, j_m)) \in \Lambda \\ 0 & \text{otherwise,} \end{cases}$$

   where $a \in \mathcal{A}$ is arbitrary and $(i_1, j_1) \neq \cdots \neq (i_m, j_m)$.

Definition 3.2. We say that $(\mathcal{A}_{i,j})$ is *matricially free* with respect to $(\varphi_{i,j})$ if

1. for any $a_k \in \operatorname{Ker}\varphi_{i_k,j_k} \cap \mathcal{A}_{i_k,j_k}$, where $k \in [m]$ and $(i_1, j_1) \neq \cdots \neq (i_m, j_m)$, and for any diagonal state φ, it holds that

$$\varphi(a_1 a_2 \ldots a_m) = 0,$$

2. $(1_{i,j})$ is a matricially free array of units associated with $(\mathcal{A}_{i,j})$ and $(\varphi_{i,j})$.

Definition 3.3. The array of variables $(a_{i,j})$ in a unital algebra ($*$-algebra) $\mathcal{A}$ will be called *matricially free* (*$*$-matricially free*) with respect to $(\varphi_{i,j})$ if there exists an array of elements (projections) $(1_{i,j})$ which is a matricially free array of units associated with $\mathcal{A}$ and $(\varphi_{i,j})$ and such that the array of algebras ($*$-algebras) generated by $a_{i,j}$ and $1_{i,j}$, respectively, is matricially free with respect to $(\varphi_{i,j})$.

Slightly less general was the setting given in [5], where we assumed that all diagonal states coincide with some distinguished state φ and the off-diagonals states agree with a family of additional states $(\varphi_j)_{j\in I}$, called *conjugate states* (or *conditions*, as in [5]), associated with φ, which are defined by

$$\varphi_j(a) = \varphi(c_j a b_j)$$

for some $c_j, b_j \in \mathcal{A}_{j,j} \cap \operatorname{Ker}\varphi$ such that $\varphi(c_j b_j) = 1$ (if $\mathcal{A}$ is a $*$-algebra, $c_j = b_j^*$). In this setting, we assume that φ and φ_j's are normalized according to

$$\varphi(1_{i,j}) = \delta_{i,j} \quad \text{and} \quad \varphi_j(1_{i,k}) = \delta_{j,k}$$

for any i, j, k. However, in this paper we will also use other arrays, for instance such in which all states in the j-th column agree with some φ_j, where $(\varphi_j)_{j\in I}$ is a given family of states on $\mathcal{A}$ which are not a priori assumed to be conjugate states associated with some distinguished state. This motivates the following definition.

Definition 3.4. Let φ and φ_j, $j \in I$, be states on $\mathcal{A}$ and let $(\varphi_{i,j})$, where $(i,j) \in J$ and $\Delta \subseteq J \subseteq I \times I$, be an array of states on $\mathcal{A}$. If $\varphi_{j,j} = \varphi$ and $\varphi_{i,j} = \varphi_j$ for any $(i,j) \in J$, we will say that $(\varphi_{i,j})$ is *defined by φ and the family $(\varphi_j)_{j\in I}$*. In turn, if $\varphi_{i,j} = \varphi_j$ for any $(i,j) \in J$, we will say that $(\varphi_{i,j})$ is *defined by the family $(\varphi_j)_{j\in I}$*.

In this context, let us remark that a Hilbert space setting, similar to that in [5], can be given for a family $(\varphi_j)_{j\in I}$ of product states on the unital $*$-algebra

$$\mathcal{A} := \bigsqcup_{i,j} \mathcal{A}_{i,j}$$

which are not defined as conjugate states associated with some distinguished state on $\mathcal{A}$. Here, $\bigsqcup_{i,j} \mathcal{A}_{i,j}$ stands for the free product of C^*-algebras without identification of units which is assumed to contain the empty word playing the role of the algebra unit.

In this 'tracial' framework, if $(\mathcal{H}_{i,j}, \pi_{i,j}, \xi_{i,j})$ is the array of GNS triples associated with an array $(\mathcal{A}_{i,j}, \varphi_{i,j})$ of noncommutative probability spaces, the underlying product Hilbert space is of the form

$$\mathcal{H} = \bigoplus_{m=1}^{\infty} \bigoplus_{(j_1,j_2)\neq\cdots\neq(j_{m-1},j_m)\neq(j_m,j_m)} \mathcal{H}^0_{j_1,j_2} \otimes \cdots \otimes \mathcal{H}^0_{j_{m-1},j_m} \otimes \mathcal{H}_{j_m,j_m},$$

where $\mathcal{H}^0_{i,j}$ is the orthocomplement of $\mathbb{C}\xi_{i,j}$ for any $(i,j) \in J$ and the last (diagonal) Hilbert space in each tensor product is $\mathcal{H}_{j_m,j_m} = \mathcal{H}^0_{j_m,j_m} \oplus \mathbb{C}\xi_{j_m,j_m}$. The space $\mathcal{H}$ is endowed with the canonical inner product and replaces the matricially free product of Hilbert spaces [5].

Nevertheless, $\mathcal{H}$ can be embedded in a larger matricially free product of Hilbert spaces, namely $(\mathcal{G}, \eta) = *_{i,j}^{M}(\mathcal{G}_{i,j}, \eta_{i,j})$, where

$$\mathcal{G}_{i,j} = \begin{cases} \mathcal{H}_{i,j} & \text{if } i \neq j \\ \mathcal{H}_{j,j} \oplus \mathbb{C}\eta_{j,j} & \text{if } i = j \end{cases} \quad \text{and} \quad \eta_{i,j} = \begin{cases} \xi_{i,j} & \text{if } i \neq j \\ \eta_{j,j} & \text{if } i = j, \end{cases}$$

with $\eta_{j,j}$ being an additional unit vector for each $j \in I$. In order to define appropriate product states on $\mathcal{A}$ associated with vectors $\xi_{j,j}$, we trivially extend each cyclic representation $\pi_{j,j} : \mathcal{A}_{j,j} \to B(\mathcal{H}_{j,j})$ to a non-cyclic representation $\gamma_{j,j} : \mathcal{A}_{j,j} \to B(\mathcal{G}_{j,j})$, keeping the off-diagonal representations unchanged, namely $\gamma_{i,j} = \pi_{i,j}$ for $i \neq j$.

Then, we take the matricially free product $\gamma = *_{i,j}^{M}\gamma_{i,j}$ and observe that $\mathcal{H}$ is left invariant by $\gamma(a)$ for any $a \in \mathcal{A}$. This allows us to define the unital $*$-representation

$$\lambda : \mathcal{A} \to B(\mathcal{H}) \quad \text{as} \quad \lambda = \gamma|\mathcal{H}$$

which then leads to product states φ_j on $\mathcal{A}$ defined by

$$\varphi_j(a) = \langle \lambda(a)\xi_{j,j}, \xi_{j,j} \rangle$$

where $j \in I$. An equivalent formulation in terms of partial isometries leading to the product states φ_j can also be given.

The proposition given below provides the main motivation for allowing a more general class of arrays in the definition of matricial freeness.

Proposition 3.1. *The array of C^*-algebras $(\mathcal{A}_{i,j})$ viewed as $*$-subalgebras of $\mathcal{A} = \bigsqcup_{i,j} \mathcal{A}_{i,j}$ is matricially free with respect to the array $(\psi_{i,j})$ of states on $\mathcal{A}$ defined by the family $(\varphi_j)_{j \in I}$ introduced above.*

Proof. In fact, it can be seen that an analog of [5, Proposition 2.3] holds for the states φ_j defined above. However, in this case it suffices to take $a_k \in \mathcal{A}_{i_k,j_k}$, where $k \in [n]$ and $(i_1, j_1) \neq \cdots \neq (i_n, j_n)$ (thus, we do not need to assume that $(i_1, j_1) \neq (j, j) \neq (i_n, j_n)$). If $a_k \in \operatorname{Ker} \varphi_{i_k,j_k}$ for $k \in [n]$, then

$$\varphi_j(a_1 a_2 \ldots a_n) = 0$$

for each j. Moreover, if $a_r = 1_{i_r,j_r}$ and $a_m \in \operatorname{Ker} \varphi_{i_m,j_m}$ for $r < m \leqslant n$, then

$$\varphi_j(a_1 \ldots a_n) = \begin{cases} \varphi_j(a_1 \ldots a_{r-1} a_{r+1} \ldots a_n) & \text{if } ((i_r, j_r), \ldots, (i_n, j_n)) \in \Lambda \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for any $a \in \mathcal{A}$, $u_1, u_2 \in \mathcal{I}$ and $i, j, k \in I$, it holds that

$$\varphi_j(u_1 a u_2) = \varphi_j(u_1)\varphi_j(a)\varphi_j(u_2) \quad \text{and} \quad \varphi_j(1_{i,k}) = \delta_{j,k}.$$

All these facts imply that $(\mathcal{A}_{i,j})$ is matricially free with respect to the array $(\psi_{i,j})$, where $\psi_{i,j} = \varphi_j$ for any i, j. $\square$

Remark 3.1. In this context, let us observe that if $(\mathcal{A}_{i,j})$ is matricially free with respect to the array defined by φ and the associated conjugate states $(\varphi_j)_{j\in I}$, then $(\mathcal{A}_{i,j})$ is not, in general, matricially free with respect to the array defined by $(\varphi_j)_{j\in I}$ since condition (1) of Definition 3.2 does not need to hold if φ is replaced by φ_j and we take $a_1, a_m \in \mathcal{A}_{j,j}$.

Let us assume now that for any natural n we have an n-dimensional square array of self-adjoint variables $(X_{i,j}(n))$ in a unital $*$-algebra $\mathcal{A}(n)$ equipped with an array of states $(\phi_{i,j}(n))$ defined by a family of states $(\phi_j(n))_{1\leqslant j\leqslant n}$. Blocks are defined by partitioning the set $\{1,2,\ldots,n\}$ into disjoint non-empty subsets,

$$[n] = N_1 \cup N_2 \cup \cdots \cup N_r,$$

where $r \in \mathbb{N}$, with dependence on n suppressed in the notation, whose sizes increase as $n \to \infty$ so that $n_j/n \to d_j$, where n_j is the cardinality of N_j for any $j \in [r]$. Then the numbers d_j form a diagonal matrix D of trace one called the *dimension matrix*.

Concerning the distributions of the considered arrays, we assume that

(A1) $(X_{i,j}(n))$ is matricially free with respect to $(\phi_{i,j}(n))$ for any $n \in \mathbb{N}$,

(A2) the variables have zero expectations,

$$\phi_{i,j}(n)\big(X_{i,j}(n)\big) = 0$$

for all $i, j \in [n]$ and $n \in \mathbb{N}$,

(A3) their variances are block-identical and are of order $1/n$, namely

$$\phi_{i,j}(n)\big(X_{i,j}^2(n)\big) = \frac{u_{p,q}}{n}$$

for any $i \in N_p$, $j \in N_q$, where each $u_{p,q}$ is a non-negative real number,

(A4) their moments are uniformly bounded, i.e. $\forall m\ \exists M_m \geqslant 0$ such that

$$\big|\phi_{i,j}(n)\big(X_{i,j}^m(n)\big)\big| \leqslant \frac{M_m}{n^{m/2}}$$

for all $i, j \in [n]$ and $n \in \mathbb{N}$.

In particular, if the distributions of the $X_{i,j}(n)$'s in the states $\phi_{i,j}(n)$ are block-identical, assumptions (A3)–(A4) are satisfied.

Following [5], where we studied limit distributions of random pseudomatrices under $\psi(n)$, we consider now normalized partial traces

$$\psi_k(n) = \frac{1}{n_k}\sum_{j\in N_k}\phi_j(n),$$

where $k \in [r]$ and $n \in \mathbb{N}$. Easy modifications of the proofs given in [5, Lemma 6.1 and Theorem 6.1] lead to combinatorial formulas for the limit distributions under partial traces given above.

Theorem 3.1. *(See [5].) Under assumptions* (A1)–(A4)*, the limit distributions of random pseudomatrices under partial traces have the form*

$$\lim_{n\to\infty}\psi_k(n)\big(S^m(n)\big)=\sum_{\pi\in\mathcal{NC}_m^2}b_k(\pi),$$

where $k\in[r]$, $m\in\mathbb{N}$ and $B=DU$, with D being the dimension matrix. Consequently, $\psi(n)(S^m(n))$ converges to $\sum_{\pi\in\mathcal{NC}_m^2}b(\pi)$ as $n\to\infty$, where $b(\pi)=\sum_{k=1}^r d_k b_k(\pi)$.

The above result refers to the 'tracial' framework which reminds the limit theorem for random matrices and is of main interest to us in this work. However, we have also shown in [5] that a similar result holds for the 'standard' framework which reminds the central limit theorem and involves the distributions of random pseudomatrices in the distinguished states $\phi(n)$. This can be phrased as follows.

Theorem 3.2. *(See [5].) Under assumptions* (A1)–(A4) *for arrays of states $(\phi_{i,j}(n))$ defined by a distinguished state $\phi(n)$ and the associated conjugate states $(\phi_j(n))_{1\leqslant j\leqslant n}$ for each n,*

$$\lim_{n\to\infty}\phi(n)\big(S^m(n)\big)=\sum_{\pi\in\mathcal{NC}_m^2}b_0(\pi)$$

where $m\in\mathbb{N}$ and $B=DU$.

In the sequel we will derive Hilbert space realizations of the limit joint distributions of random pseudomatrices $S(n)$ and their blocks under $\phi(n)$, $\psi_k(n)$ and $\psi(n)$. Interestingly enough, all these realizations are given on the same type of Hilbert space which is a matricially free product of Fock spaces.

4. Matricially free Gaussian operators

In this section we shall introduce self-adjoint operators which play the role of matricially free Gaussian operators living in the matricially free product of Fock spaces.

Recall that by the boolean and free Fock spaces over the Hilbert space $\mathcal{H}$, respectively, we understand the direct sums

$$\mathcal{F}_0(\mathcal{H})=\mathbb{C}\xi\oplus\mathcal{H}\quad\text{and}\quad\mathcal{F}(\mathcal{H})=\mathbb{C}\xi\oplus\bigoplus_{m=1}^{\infty}\mathcal{H}^{\otimes m},$$

where ξ is a unit vector, endowed with the canonical inner products. We shall use them to define an array of Fock spaces and their matricially free product.

Definition 4.1. Let $(\mathcal{H}_{i,j},\xi_{i,j})$ be an array of Hilbert spaces with distinguished unit vectors. By the *matricially free product of* $(\mathcal{H}_{i,j},\xi_{i,j})$ we understand the pair $(\mathcal{H},\xi)$, where

$$\mathcal{H}=\mathbb{C}\xi\oplus\bigoplus_{m=1}^{\infty}\bigoplus_{(i_1,i_2)\neq\cdots\neq(i_m,i_m)}\mathcal{H}^0_{i_1,i_2}\otimes\mathcal{H}^0_{i_2,i_3}\otimes\cdots\otimes\mathcal{H}^0_{i_m,i_m},$$

with $\mathcal{H}_{i,j}^{0}=\mathcal{H}_{i,j}\ominus\mathbb{C}\xi_{i,j}$ and ξ being a unit vector, with the canonical inner product. We denote it $(\mathcal{H},\xi)=*_{i,j}^{M}(\mathcal{H}_{i,j},\xi_{i,j})$.

We already know that the matricially free Fock space, which is a matricially free analog of the free Fock space, is the matricially free product of an array of free Fock spaces [5]. Nevertheless, in order to find Hilbert space realizations of the limit laws of Theorems 3.1 and 3.2, it suffices to take a matricially free product of an array of Fock spaces, in which free Fock spaces are put only on the diagonal and the boolean Fock spaces elsewhere. Clearly, we obtain a truncation of the matricially free Fock space in this fashion. This structure reflects the difference between diagonal and off-diagonal random variables and is related to the difference between multiplication of diagonal and off-diagonal blocks of usual matrices.

Definition 4.2. By the *matricially free-boolean Fock space* over the array $\hat{\mathcal{H}}=(\mathcal{H}_{i,j})$ we shall understand the matricially free product

$$(\mathcal{F},\xi)=*_{i,j}^{M}(\mathcal{F}_{i,j},\xi_{i,j}),\quad \text{where}\quad \mathcal{F}_{i,j}=\begin{cases}\mathcal{F}(\mathcal{H}_{j,j}) & \text{if } i=j\\ \mathcal{F}_0(\mathcal{H}_{i,j}) & \text{if } i\neq j\end{cases}$$

and $\xi_{i,j}$ denotes the distinguished unit vector in $\mathcal{F}_{i,j}$.

Remark 4.1. If we have a square array $\hat{\mathcal{H}}=(\mathcal{H}_{i,j})$, where $\mathcal{H}_{i,j}\cong\mathcal{H}_i$ for any $i,j\in I$ and $(\mathcal{H}_i)$ is a family of Hilbert spaces, then we have a natural isomorphism

$$\mathcal{F}\cong\mathcal{F}\Big(\bigoplus_{j}\mathcal{H}_j\Big)$$

since each tensor product

$$\big(\mathcal{H}_{i_1,i_1}^{\otimes(n_1-1)}\otimes\mathcal{H}_{i_1,i_2}\otimes\mathcal{H}_{i_2,i_2}^{\otimes(n_2-1)}\big)\otimes\cdots\otimes\big(\mathcal{H}_{i_{m-1},i_{m-1}}^{\otimes(n_{m-1}-1)}\otimes\mathcal{H}_{i_{m-1},i_m}\otimes\mathcal{H}_{i_m,i_m}^{\otimes n_m}\big)$$

is isomorphic to

$$\mathcal{H}_{i_1}^{\otimes n_1}\otimes\mathcal{H}_{i_2}^{\otimes n_2}\otimes\cdots\otimes\mathcal{H}_{i_m}^{\otimes n_m}$$

for any $i_1\neq i_2\neq\cdots\neq i_m$ and $m,n_1,n_2,\ldots,n_m\in\mathbb{N}$. Thus, $\mathcal{F}$ is in this case also isomorphic to the strongly matricially free Fock space $\mathcal{R}(\hat{\mathcal{H}})$ introduced in [5].

Example 4.1. A simple example of a matricially free-boolean Fock space is the matricially free product of the two-dimensional array of the form

$$(\mathcal{F}_{i,j})=\begin{pmatrix} l_2(G_{1,1}) & l_2(G_{1,2})\\ l_2(G_{2,1}) & l_2(G_{2,2})\end{pmatrix},$$

where $G_{j,j}=FS(g_{j,j})$, the free semigroup on one generator $g_{j,j}$, where $j\in\{1,2\}$, and $G_{i,j}=\mathbb{Z}_2$ with the generator denoted $g_{i,j}$ for $i\neq j$. Then $\mathcal{F}=l_2(G)$, where G is the 'matricially free product of semigroups' $G_{i,j}$, by which we understand the subset of their free product $*_{i,j}G_{i,j}$ given by the union

$$G = \bigcup_{n=0}^{\infty} G^{(n)}$$

of disjoint subsets, where $G^{(n)}$ consists of words of type $g_1 g_2 \ldots g_n$ with g_k being an element of $G^0_{i_k,j_k}$, where $G^0_{i,j} := G_{i,j} \setminus \{\epsilon_{i,j}\}$, where $\epsilon_{i,j}$ is the unit in $G_{i,j}$, with $((i_1, j_1), \ldots, (i_n, j_n)) \in \Lambda$ and $i_n = j_n$. For instance,

$$\begin{aligned}
&G^{(0)} = \{e\}, \\
&G^{(1)} = \{g^k_{1,1}, g^k_{2,2}\colon k \in \mathbb{N}\}, \\
&G^{(2)} = \{g_{2,1} g^k_{1,1}, g_{1,2} g^k_{2,2}\colon k \in \mathbb{N}\}, \\
&G^{(3)} = \{g^k_{2,2} g_{2,1} g^m_{1,1}, g^k_{1,1} g_{1,2} g^m_{2,2}, g_{1,2} g_{2,1} g^m_{1,1}, g_{2,1} g_{1,2} g^m_{2,2}\colon k, m \in \mathbb{N}\},
\end{aligned}$$

etc., with the remaining subsets consisting of words built from 'matricially free products' of powers of the generators, where the diagonal generators admit all natural powers, whereas the off-diagonal ones admit only the powers equal to one.

Definition 4.3. Let $A = (\alpha_{i,j})$ be a diagonal-containing array of positive real numbers and let $(\mathcal{H}_{i,j}) = (\mathbb{C}e_{i,j})$ be the associated array of Hilbert spaces. By the *matricially free creation operators* associated with A we understand operators of the form

$$\varsigma_{i,j} = \alpha_{i,j} \tau^* \ell(e_{i,j}) \tau,$$

where $\tau : \mathcal{F} \to \mathcal{F}(\bigoplus_{i,j} \mathcal{H}_{i,j})$ is the canonical embedding and the $\ell(e_{i,j})$'s denote the canonical free creation operators. By the *matricially free annihilation operators* and the *matricially free Gaussian operators* we understand their adjoints $\varsigma^*_{i,j}$ and sums denoted $\zeta_{i,j} = \varsigma_{i,j} + \varsigma^*_{i,j}$, respectively.

We shall assume now that A is a diagonal-containing subarray of a finite square array and that it is indexed by the set J (thus, $\Delta \subseteq J \subseteq I \times I$). Moreover, it is convenient to assume that A is the square root of another array B taken entrywise, i.e.

$$\alpha_{i,j} = \sqrt{b_{i,j}}, \quad \text{written} \quad A = \sqrt{B},$$

where B can be assumed to be a subarray of a square array. The dependence on A of the operators defined above is suppressed in our notations.

We shall prove below that the $*$-algebras $\mathcal{A}_{i,j}$, each generated by $\varsigma_{i,j}$ and suitably defined unit $1_{i,j}$, respectively, are matricially free with respect to a suitably defined array of states on $B(\mathcal{F})$. Namely, $1_{i,j}$ is defined as the projection onto the subspace of $\mathcal{F}$ onto which the $*$-algebra generated by the creation operator $\varsigma_{i,j}$ acts non-trivially. To be more precise, let us introduce projections $s_{i,j}$ and $r_{i,j}$ for any $(i,j) \in J$ which give an orthogonal decomposition of $1_{i,j}$, i.e. $1_{i,j} = r_{i,j} + s_{i,j}$ and $r_{i,j} s_{i,j} = 0$. Namely, $s_{i,j}$ is the canonical projection onto the subspace of $\mathcal{F}$ spanned by tensors which begin with $e_{i,j}$ for any $(i,j) \in J$. In turn, $r_{i,j}$ is the canonical projection onto the subspace of $\mathcal{F}$ spanned by tensors which begin with $e_{j,k}$ for some k such that $(j,k) \in J$ if $i \neq j$, whereas $r_{j,j}$ is the canonical projection onto the subspace spanned by the vacuum vector and tensors which begin with $e_{j,k}$ for $k \neq j$, where $(j,k) \in J$.

Two types of transformations on the considered operators will be performed: truncations and symmetrizations. Here, we shall introduce *truncated matricially free creation operators* and *truncated units* by

$$\wp_{i,j} = \varsigma_{i,j} P \quad \text{and} \quad t_{i,j} = 1_{i,j} P,$$

respectively, where P is the canonical projection onto $\mathcal{F} \ominus \mathbb{C}\Omega$. The *truncated matricially free annihilation operators* and *truncated matricially free Gaussian operators*, respectively, will be denoted by $\wp_{i,j}^*$ and $\omega_{i,j} = \wp_{i,j} + \wp_{i,j}^*$ for any i, j.

Finally, in the case of finite-dimensional arrays, which will be considered from now on, we use the same symbol to denote the sum of operators in a given array, like for instance sums of creation and annihilation operators will be denoted

$$\varsigma = \sum_{(i,j) \in J} \varsigma_{i,j} \quad \text{and} \quad \varsigma^* = \sum_{(i,j) \in J} \varsigma_{i,j}^*,$$

respectively, their truncations, $\wp$ and $\wp^*$, and the sums of Gaussians operators and their truncations,

$$\zeta = \sum_{(i,j) \in J} \zeta_{i,j} \quad \text{and} \quad \omega = \sum_{(i,j) \in J} \omega_{i,j},$$

called the *Gaussian pseudomatrix* and the *truncated Gaussian pseudomatrix*, respectively. Note that all these operators are bounded since the considered sums are finite.

Proposition 4.1. *Let $(\Phi_{i,j})$ be the array of states on $B(\mathcal{F})$ defined by the vacuum state Φ and the family $(\Psi_j)_{j \in I}$, where Ψ_j is the vector state associated with $e_{j,j}$ for any $j \in I$. Then*

1. *the Φ-distribution of $\zeta_{j,j}$ is the semicircle law of radius $2\alpha_{j,j}$ for any j,*
2. *the Ψ_j-distribution of $\zeta_{i,j}$ is the Bernoulli law concentrated at $\pm\alpha_{i,j}$ for any $i \neq j$,*
3. *the array $(\zeta_{i,j})$ is matricially free with respect to $(\Phi_{i,j})$.*

Proof. The first two claims follow easily from the definitions of the operators involved. Next, instead of proving the third claim, we will prove a slightly more general result that the array $(\mathcal{A}_{i,j})$, where $\mathcal{A}_{i,j} = \mathbb{C}\langle \varsigma_{i,j}, \varsigma_{i,j}^*, 1_{i,j} \rangle$ for any $(i, j) \in J$, is matricially free with respect to $(\Phi_{i,j})$. Essentially, we proceed as in the free case [11], but we have slightly more complicated relations between the operators involved (see also Proposition 4.2 of [5]). In particular, one has to treat diagonal and off-diagonal subalgebras separately. Let us consider first the off-diagonal case, when the creation and annihilation operators satisfy relations

$$\varsigma_{i,j}^* \varsigma_{i,j} = b_{i,j} r_{i,j} \quad \text{and} \quad \varsigma_{i,j}^2 = 0, \qquad \varsigma_{i,j}^{*2} = 0$$

for any $i \neq j$. In that case we have additional relations

$$r_{i,j} \varsigma_{i,j} = 0, \qquad \varsigma_{i,j} r_{i,j} = \varsigma_{i,j}, \qquad s_{i,j} \varsigma_{i,j} = \varsigma_{i,j}, \qquad \varsigma_{i,j} s_{i,j} = 0.$$

Using all these relations and their adjoints, as well as equations

$$\Psi_j(r_{i,j}) = 1, \qquad \Psi_j(s_{i,j}) = 0,$$

we deduce that an arbitrary noncommutative polynomial from $\mathcal{A}_{i,j} \cap \mathrm{Ker}(\Psi_j)$, where $i \neq j$, is spanned by $\varsigma_{i,j}$, $\varsigma_{i,j}^*$, and $s_{i,j}$. In the diagonal case, the situation is similar to that in the free case since each pair of diagonal creation and annihilation operators satisfies the relation

$$\varsigma_{j,j}^* \varsigma_{j,j} = b_{j,j} 1_{j,j} \quad \text{with} \quad \Phi(1_{j,j}) = 1,$$

and therefore any polynomial from $\mathcal{A}_{j,j} \cap \mathrm{Ker}(\Phi)$ is spanned by $1_{j,j}$ and $\varsigma_{j,j}^p \varsigma_{j,j}^{*q}$, where $p + q > 0$, for any j. Moreover, $s_{i,j}\varsigma_{k,l} = 0$ and $\varsigma_{k,l} s_{i,j} = \delta_{l,i}\varsigma_{k,l}$ for any $(i, j) \neq (k, l)$. Hence, to prove matricial freeness of the array $(\mathcal{A}_{i,j})$ with respect to $(\Phi_{i,j})$, it suffices to show that

$$\Phi\big(\varsigma_{i_1,j_1}^{p_1} \varsigma_{i_1,j_1}^{*q_1} \cdots \varsigma_{i_m,j_m}^{p_m} \varsigma_{i_m,j_m}^{*q_m}\big) = 0$$

for suitable powers $p_1, q_1, \ldots, p_m, q_m$ that depend on whether the corresponding operators are diagonal or not, since $\Phi(s_{i_1,j_1} \ldots s_{i_m,j_m}) = 0$ for any off-diagonal pairs $(i_1, j_1) \neq \cdots \neq (i_m, j_m)$. At this point we can use the same inductive argument as in the free case [11], which gives the above 'freeness condition'. Moreover, the definition of each $1_{i,j}$ shows that it is the projection onto the subspace onto which the $*$-algebra generated by $\varsigma_{i,j}$ acts non-trivially, which implies that the array $(1_{i,j})$ is the matricially free array of units. □

Note that in Proposition 4.1 each Ψ_j is a conjugate state associated with Φ since there exists $b_j \in \mathcal{A}_{j,j} \cap \mathrm{Ker}\,\Phi$, namely $b_j = \varsigma_{j,j}/\alpha_{j,j}$, such that $\Psi_j(a) = \Phi(b_j^* a b_j)$ for any $a \in B(\mathcal{F})$. However, a similar result is obtained for arrays of states defined by the family $(\Psi_j)_{j\in I}$, where Ψ_j is now the vector state on $B(\mathcal{F} \ominus \mathbb{C}\Omega)$ associated with $e_{j,j}$ for any $j \in I$.

Proposition 4.2. *Let $(\Psi_{i,j})$ be the array of states on $B(\mathcal{F} \ominus \mathbb{C}\Omega)$ defined by the family $(\Psi_j)_{j\in I}$, where Ψ_j is the vector state associated with $e_{j,j}$ for any j. Then*

1. *the Ψ_j-distribution of $\omega_{j,j}$ is the semicircle law of radius $2\alpha_{j,j}$ for any j,*
2. *the Ψ_j-distribution of $\omega_{i,j}$ is the Bernoulli law concentrated at $\pm\alpha_{i,j}$ for any $i \neq j$,*
3. *the array $(\omega_{i,j})$ is matricially free with respect to $(\Psi_{i,j})$.*

Proof. The proof is similar to that of Proposition 4.1. □

5. Fock-space realizations of limit distributions

Using the matricially free Gaussian operators and their truncations, we will find realizations on $\mathcal{F}$ of the limit distributions of Theorems 3.1–3.2.

To each $\pi \in \mathcal{NC}_m^2$ for m even we assign natural products of creation and annihilation operators. First, to each $\pi \in \mathcal{NC}_m^2$ we assign the sequence $\epsilon(\pi) = (\epsilon_1, \epsilon_2, \ldots, \epsilon_m)$, where $\epsilon_j = 1$ whenever $j \in \mathcal{R}(\pi)$ and $\epsilon_k = *$ whenever $k \in \mathcal{L}(\pi)$. Then, to each $\pi \in \mathcal{NC}_m^2$ we assign products of creation and annihilation operators

$$\varsigma(\pi)=\varsigma^{\epsilon_1}\varsigma^{\epsilon_2}\ldots\varsigma^{\epsilon_m}\quad\text{and}\quad\wp_j(\pi)=\wp^{\epsilon_1}\wp^{\epsilon_2}\ldots\wp^{\epsilon_m}$$

for any $j\in[r]$, where $(\epsilon_1,\epsilon_2,\ldots,\epsilon_m)=\epsilon(\pi)$.

There is a nice relation between the expectations of these products and numbers $b_j(\pi)$ defined in Section 2. In the case of $b_0(\pi)$ we will use the expectations in the vacuum state Φ and for the remaining j's we take Ψ_j's associated with vectors $e_{j,j}$. Finally, to obtain $b(\pi)$, we will use the convex linear combination of states of the form

$$\Psi=\sum_{j=1}^{r}d_j\Psi_j,$$

where numbers $d_1,d_2,\ldots,d_r$ are taken from the dimension matrix D.

We are ready to give Fock-space realizations of the limit distributions of random pseudomatrices in terms of the distributions of Gaussian and truncated Gaussian pseudomatrices.

Lemma 5.1. *For the limit distributions of Theorem* 3.2*, it holds that*

$$\sum_{\pi\in\mathcal{NC}_m^2}b_0(\pi)=\Phi\big(\zeta^m\big)$$

for any $m\in\mathbb{N}$, where ζ is the Gaussian pseudomatrix.

Proof. If m is odd, both sides of the first equation are clearly zero. Therefore, suppose that m is even. We have

$$\Phi\big(\zeta^m\big)=\sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_m\in\{1,*\}}\Phi\big(\varsigma^{\epsilon_1}\varsigma^{\epsilon_2}\ldots\varsigma^{\epsilon_m}\big)=\sum_{\pi\in\mathcal{NC}_m^2}\Phi\big(\varsigma(\pi)\big),$$

where we used the fact that there is a bijection between non-vanishing moments of type $\Phi(\varsigma^{\epsilon_1}\varsigma^{\epsilon_2}\ldots\varsigma^{\epsilon_m})$ and non-crossing pair partitions $\mathcal{NC}_m^2$ and thus to each tuple $(\epsilon_1,\ldots,\epsilon_m)$ associated with a non-vanishing moment there corresponds a unique $\pi\in\mathcal{NC}_m^2$ such that $(\epsilon_1,\ldots,\epsilon_m)=\epsilon(\pi)$. Essentially, this fact follows from the definition of the creation operators $\varsigma_{i,j}^{\epsilon}$ as truncated free creation operators. Thus, the assertion of the lemma will follow from the combinatorial formula

$$b_0(\pi)=\Phi\big(\varsigma(\pi)\big)$$

for each $\pi\in\mathcal{NC}_m^2$. Now, if $\pi\in\mathcal{NC}_{m,0}^2(p_1,q_1,\ldots,p_m,q_m)$ for some $p_1,q_1,\ldots,p_m,q_m\in[r]$, then we claim that

$$\Phi\big(\varsigma_{p_1,q_1}^{\epsilon_1}\ldots\varsigma_{p_m,q_m}^{\epsilon_m}\big)=b_0(\pi,f),$$

where $(\epsilon_1,\ldots,\epsilon_m)=\epsilon(\pi)$ and f is the coloring of π defined by p_k, where $k\in\mathcal{L}(\pi)$. This claim can be proved by induction. Suppose that the last annihilation operator in this mixed moment is indexed by k, i.e. $\epsilon_k=*$ and $\epsilon_l=1$ for $l>k$. Then $\{k,k+1\}$ must be a block which does not have any inner blocks. The corresponding product of operators is of the form $\varsigma_{p_k,q_k}^{*}\varsigma_{p_k,q_k}$. Two cases are possible:

1. If $k = m - 1$ and $p_k = q_k$, then $\varsigma^*_{p_k,q_k}\varsigma_{p_k,q_k}$ acts on Ω and gives $b_{p_k,q_k}\Omega$.
2. If $k < m - 1$ and $q_{k+1} = p_{k+2}$, then $\varsigma^*_{p_k,q_k}\varsigma_{p_k,q_k}$ acts on some simple tensor h and gives $b_{p_k,q_k}h$. Here, q_k colors the nearest outer block of $\{k, k+1\}$.

If we repeat this procedure for the product of operators corresponding to the partition π' obtained from π by removing block $\{k, k+1\}$, we obtain the product of $b_{p,q}$'s which appears in the combinatorial formula for $b_0(\pi, f)$ after a finite number of steps. If we fix π and sum over all tuples $(p_1, q_1, \ldots, p_m, q_m)$ for which $\pi \in \mathcal{NC}_{m,0}(p_1, q_1, \ldots, p_m, q_m)$ (Definition 2.2), we obtain in fact the summation over p_k, where $k \in \mathcal{L}(\pi)$, equivalent to the summation over $F_r(\pi)$, which proves the formula for $b_0(\pi)$ since mixed moments corresponding to the remaining tuples are equal to zero. This completes the proof. □

Lemma 5.2. *For the limit distributions of Theorem* 3.1*, it holds that*

$$\sum_{\pi\in\mathcal{NC}^2_m} b_k(\pi) = \Psi_k\big(\omega^m\big)$$

for $k \in [r]$ and $m \in \mathbb{N}$, and hence $\sum_{\pi\in\mathcal{NC}^2_m} b(\pi) = \Psi(\omega^m)$, where ω is the truncated Gaussian pseudomatrix.

Proof. The proof is similar to that of Lemma 5.1 and is based on the analogous combinatorial formula

$$b_k(\pi) = \Psi_k\big(\wp(\pi)\big)$$

for any $1 \leqslant k \leqslant r$ and $\pi \in \mathcal{NC}^2_m$, where m is even and positive. The proof of that formula reduces to showing that if $(\epsilon_1, \ldots, \epsilon_m) = \epsilon(\pi)$ for some $\pi \in \mathcal{NC}^2_m(p_1, q_1, \ldots, p_m, q_m)$ and $k = q_m$, then

$$\Psi_k\big(\wp^{\epsilon_1}_{p_1,q_1}\cdots\wp^{\epsilon_m}_{p_m,q_m}\big) = b_k(\pi, f),$$

where f is the coloring of π defined by p_j, where $j \in \mathcal{L}(\pi)$, with k coloring the imaginary block. As compared with the proof for Φ, instead of acting on Ω, we need to act on $e_{k,k}$. However, thanks to the projection P onto $\mathcal{F} \ominus \mathbb{C}\Omega$ which appears in the definition of $\wp^\epsilon_{k,k}$, each $e_{k,k}$ plays the role of the vacuum vector with respect to $\wp^\epsilon_{i,k}$ for any i, k, including the case when $i = k$ since $\wp^*_{k,k}e_{k,k} = 0$. Therefore, the arguments are similar to those for Φ, except that $\wp_{i,k}e_{k,k}$ is non-zero for arbitrary i. In terms of diagrams, this means that each covering block of π contributes $b_{i,k}$ for various i's. Finally, when we use the definition of Ψ and sum over $k \in [r]$, we obtain for each k the extra factor d_k which appears in the combinatorial formula for $b(\pi)$. Consequently, $b(\pi) = \Psi(\wp(\pi))$ for any such π, which then leads to the formula for $\Psi(\omega^n)$. □

Example 5.1. Consider the moment corresponding to the partition $\pi \in \mathcal{NC}^2_4$ consisting of two blocks: $\{1, 4\}$ and $\{2, 3\}$. Let $A = \sqrt{B} \in M_2(\mathbb{R})$ be the square root taken entrywise, where $B = DU$ and we set $\alpha_{1,1} = \alpha$, $\alpha_{1,2} = \beta$, $\alpha_{2,1} = \gamma$, $\alpha_{2,2} = \delta$. Then

$$b_0(\pi) = \alpha^4 + \alpha^2\gamma^2 + \delta^2\beta^2 + \delta^4.$$

On the other hand,

$$\varsigma^2\Omega = \alpha^2(e_{1,1}\otimes e_{1,1}) + \alpha\gamma(e_{2,1}\otimes e_{1,1}) \\ + \beta\delta(e_{1,2}\otimes e_{2,2}) + \delta^2(e_{2,2}\otimes e_{2,2})$$

and thus the 'symmetric' action of $(\varsigma^*)^2$ gives exactly $b_0(\pi)\Omega$.

Example 5.2. For the same partition π as in the above example we obtain

$$b(\pi) = d_1\big(\alpha^4+\alpha^2\gamma^2+\beta^2\gamma^2+\gamma^2\delta^2\big) \\ + d_2\big(\alpha^2\beta^2+\beta^2\gamma^2+\beta^2\delta^2+\delta^4\big).$$

On the other hand,

$$\Psi\big(\wp(\pi)\big) = d_1\Psi_1\big(\wp(\pi)\big) + d_2\Psi_2\big(\wp(\pi)\big),$$

where $\wp(\pi) = \wp^*\wp^*\wp\wp$, and we get

$$\wp^2 e_{1,1} = \gamma\big(\delta(e_{2,2}\otimes e_{2,1}\otimes e_{1,1}) + \beta(e_{1,2}\otimes e_{2,1}\otimes e_{1,1})\big) \\ + \alpha\big(\alpha(e_{1,1}\otimes e_{1,1}\otimes e_{1,1}) + \gamma(e_{2,1}\otimes e_{1,1}\otimes e_{1,1})\big),$$

which, in view of the 'symmetric' action of the adjoint, gives

$$d_1\Psi_1\big(\wp^*\wp^*\wp\wp\big) = d_1\big(\gamma^2\delta^2+\beta^2\gamma^2+\alpha^2\gamma^2+\alpha^4\big).$$

A similar expression is obtained for $d_2\Psi_2(\wp^*\wp^*\wp\wp)$, with α interchanged with δ and β interchanged with γ. The sum of both expressions agrees with $b(\pi)$.

6. Matricial freeness of blocks

We show in this section that blocks of finite-dimensional arrays of matricially free random variables are matricially free with respect to an appropriately defined array of states. This is the analog of the property of free random variables which says that families of sums of free random variables are free.

Definition 6.1. Suppose that the n-dimensional array $(X_{i,j})$ of variables from a unital algebra $\mathcal{A}$ is matricially free with respect to $(\varphi_{i,j})$ and let $[n] = N_1\cup N_2\cup\cdots\cup N_r$ be a partition of $[n]$ into disjoint non-empty subsets. The sums of the form

$$S_{p,q} = \sum_{(i,j)\in N_p\times N_q} X_{i,j},$$

where $p,q\in[r]$, will be called *blocks* of the pseudomatrix $S=\sum_{i,j}X_{i,j}$. We will say that the array $(X_{i,j})$ has *block-identical distributions* with respect to $(\varphi_{i,j})$ if the $\varphi_{i,j}$-distributions of $X_{i,j}$ are the same for all $(i,j)\in N_p\times N_q$ and fixed $p,q\in[r]$.

We need to define the associated array of block units which satisfy the conditions of Definition 3.1. In order to construct them on the level of noncommutative probability spaces, let us first return to the more intuitive framework of the matricially free product of representations of the array $(\mathcal{A}_{i,j})$ of unital C^*-algebras on the matricially free product of Hilbert spaces

$$(\mathcal{H},\xi) = *^{M}_{i,j}(\mathcal{H}_{i,j},\xi_{i,j})$$

and then carry over the corresponding definition to the algebraic framework of noncommutative probability spaces, where each $\mathcal{A}_{i,j}$ is equipped with an internal unit $1_{i,j}$ and a state $\varphi_{i,j}$. It is the matricially free product of Hilbert spaces on which one defines the canonical $*$-representations of $(\mathcal{A}_{i,j},\varphi_{i,j})$. Namely, let $(\mathcal{H}_{i,j},\pi_{i,j},\xi_{i,j})$ be the associated GNS triples, so that $\varphi_{i,j}(a) = \langle \pi_{i,j}(a)\xi_{i,j},\xi_{i,j}\rangle$ for any $a \in \mathcal{A}_{i,j}$. By $\lambda_{i,j}$ we denote the canonical $*$-representation of $(\mathcal{A}_{i,j},\varphi_{i,j})$ on $(\mathcal{H},\xi)$. Using these representations and appropriate partial isometries, we have defined in [5] their matricially free product $\lambda = *^{M}_{i,j}\pi_{i,j}$ which maps $\bigsqcup_{i,j}\mathcal{A}_{i,j}$ into $B(\mathcal{H})$.

Recall the notations for the so-called diagonal and off-diagonal subspaces of $\mathcal{H}$:

$$\mathcal{H}(j,j) = \mathbb{C}\xi \oplus \bigoplus_{m=2}^{\infty} \bigoplus_{\substack{(j,i_2)\neq\cdots\neq(i_m,i_m)\\ i_2\neq j}} \mathcal{H}^0_{j,i_2}\otimes\mathcal{H}^0_{i_2,i_3}\otimes\cdots\otimes\mathcal{H}^0_{i_m,i_m},$$

$$\mathcal{H}(i,j) = \bigoplus_{m=1}^{\infty} \bigoplus_{(j,i_2)\neq\cdots\neq(i_m,i_m)} \mathcal{H}^0_{j,i_2}\otimes\mathcal{H}^0_{i_2,i_3}\otimes\cdots\otimes\mathcal{H}^0_{i_m,i_m},$$

respectively, for any $i \neq j$, onto which the left free actions of the corresponding $\lambda(\mathcal{A}_{i,j})$'s are non-trivial.

In this connection note that λ does not send the units $1_{i,j}$ onto the unit of $B(\mathcal{H})$. In fact, $\lambda(1_{i,j}) = r_{i,j} + s_{i,j}$, where

$$r_{i,j} = \text{projection onto } \mathcal{H}(i,j),$$

$$s_{i,j} = \text{projection onto } \mathcal{K}(i,j),$$

and $\mathcal{K}(i,j) = \mathcal{H}^0_{i,j}\otimes\mathcal{H}(i,j)$. Clearly, $r_{i,j} \perp s_{i,j}$ for any fixed i, j and thus their sum is the canonical projection onto the subspace of $\mathcal{H}$ onto which $\lambda(\mathcal{A}_{i,j})$ acts non-trivially.

In the proposition given below we state basic properties of the projections involved that are needed for the construction of block units. Note that the case of $\mathrm{card}(I) = 1$ is trivial from the point of view of matricially free product structures and that is why it is not treated. Recall that $\mathcal{I}$ stands for the commutative unital algebra generated by the units $\lambda(1_{i,j}) \equiv 1_{i,j}$.

Proposition 6.1. *If* $\mathrm{card}(I) > 1$ *and* J_1, J_2 *are identical or disjoint finite subsets of* I, *then the algebra* $\mathbb{C}[1_{i,j}\colon\ i \in J_1,\ j \in J_2]$ *contains the canonical projection* $\mathbf{1}_{J_1,J_2}$ *onto the subspace of* $\mathcal{H}$ *onto which the algebra generated by* $\{\lambda(a_{i,j})\colon\ a_{i,j} \in \mathcal{A}_{i,j},\ i \in J_1,\ j \in J_2\}$ *acts non-trivially.*

Proof. We want to construct a projection $\mathbf{1}_{J_1,J_2}$ onto the subspace of $\mathcal{H}$ that would be suitable for the left action of $\lambda(a_{i,j})$, where $i \in J_1$ and $j \in J_2$. If $J_1 \cap J_2 = \emptyset$, then this subspace is the orthogonal direct sum

$$\mathcal{H}_{J_1,J_2} = \bigoplus_{i\in J_1, j\in J_2} \mathcal{K}(i,j) \oplus \bigoplus_{j\in J_2} \bigl(\mathcal{K}(j,j) \oplus \mathcal{H}(j,j) \ominus \mathbb{C}\xi\bigr),$$

and the associated canonical projection is

$$\mathbf{1}_{J_1,J_2} = \sum_{i\in J_1,\, j\in J_2} (1_{i,j} - 1_{j,j} + 1_{i,i}1_{j,j}) + \sum_{j\in J_2} (1_{j,j} - 1_{j,j}1_{k,k}),$$

where $k \in J_1$ is arbitrary and we have used the fact that $1_{i,i}1_{i,j} = p_\xi$ for any $i \neq j$. In turn, if $J_1 = J_2 = J$, then we obtain the orthogonal direct sum

$$\mathcal{H}_{J,J} = \mathbb{C}\xi \oplus \bigoplus_{j\in J} \bigl(\mathcal{K}(j,j) \oplus \mathcal{H}(j,j) \ominus \mathbb{C}\xi\bigr),$$

and the associated canonical projection is

$$\mathbf{1}_{J,J} = \sum_{j\in J} (1_{j,j} - 1_{j,j}1_{k,k}) + 1_{k,k}1_{l,l},$$

where indices k, l are such that $k \neq l$ and otherwise are arbitrary elements of J. This completes the proof. $\square$

The proof of the above proposition enables us to construct block units which are internal units in the commutative algebras $\mathbb{C}[S_{p,q}, \mathbf{1}_{p,q}]$, where $p, q \in [r]$, each generated by a block and the corresponding block unit. Namely, we set

$$\mathbf{1}_{p,q} := \mathbf{1}_{N_p,N_q}$$

for any $p, q \in [r]$, where the right-hand side is defined in terms of $1_{i,j}$'s in exactly the same way as in the proof of Proposition 6.1.

It will also be useful to introduce the following terminology. Namely, if

$$a_k \in \mathcal{A}_{i_k,j_k} \cap \operatorname{Ker}(\varphi_{i_k,j_k}) \quad \text{for } 1 \leqslant k \leqslant n \text{ and } \bigl((i_1,j_1),\ldots,(i_n,j_n)\bigr) \in \Lambda,$$

we will say that the product $a_1a_2\ldots a_n$ is in the *matricially free kernel form* with respect to $(\varphi_{i,j})$.

Theorem 6.1. *Let $(\varphi_{i,j})$ be the array of states on $\mathcal{A}$ defined by the family $(\varphi_j)_{1\leqslant j\leqslant n}$, and let $(\psi_{p,q})$ be defined by the family of associated normalized partial traces $(\psi_q)_{1\leqslant q\leqslant r}$. If $(X_{i,j})$ is matricially free and has block-identical distributions with respect to $(\varphi_{i,j})$, then $(S_{p,q})$ is matricially free with respect to $(\psi_{p,q})$.*

Proof. We assume that $(X_{i,j})$ is matricially free and has block-identical distributions with respect to $(\varphi_{i,j})$. Clearly, the unital algebra generated by $(\mathbf{1}_{p,q})$ is commutative. We claim that the array $(\mathbf{1}_{p,q})$ is a matricially free array of units associated with $(S_{p,q})$ and $(\psi_{p,q})$. The proof of condition (1) of Definition 3.1 for $(\psi_{p,q})$ follows easily from the same condition for $(\varphi_{i,j})$. To prove condition (2), we need to evaluate mixed moments of type

$$\varphi_j(w\mathbf{1}_{p,q}w_1w_2\dots w_m),$$

where $w_k \in \operatorname{Ker}(\psi_{p_k,q_k})$ is a polynomial in S_{p_k,q_k} for each $k \in [m]$ and $j \in [n]$ and the product $w_1w_2\dots w_m$ is in the matricially free kernel form with respect to $(\psi_{p,q})$. We shall reduce the computations to moments of type $\varphi_j(w\mathbf{1}_{i,l}v_1v_2\dots v_h)$, where $v_1v_2\dots v_h$ is in the matricially free kernel form with respect to $(\varphi_{i,j})$. For that purpose, let us express each power of S_{p_k,q_k} which appears in w_k in terms of variables which are in the kernels of the $\varphi_{i,j}$'s. This procedure, described in more detail below, is applied to $w_m, w_{m-1}, \dots, w_1$ (in that order).

Let $w(X) = \sum_{r=0}^{s} c_r X^r$ be an arbitrary polynomial and let $S_{p,q} = \sum_{(i,j)\in N_p\times N_q} X_{i,j}$. We decompose each positive power of $X_{i,j}$ which appears in $w(S_{p,q})$ as

$$X_{i,j}^n = \left(X_{i,j}^n\right)^0 + \varphi_{i,j}\left(X_{i,j}^n\right)1_{i,j},$$

where $i \in N_p$, $j \in N_q$. Using the fact that $(1_{i,j})$ is a matricially free array of units, we observe that in the computations of the mixed moments of the given type we can use, without loss of generality, polynomials of the form

$$w(S_{p,q}) = \sum_{r=1}^{s} \sum_{i_1,\dots,i_{r+1}} \sum_{n_1,\dots,n_r} c_{i_1,\dots,i_r,i_{r+1}}^{n_1,\dots,n_r} \left(X_{i_1,i_2}^{(n_1)}\right)^0 \dots \left(X_{i_r,i_{r+1}}^{(n_r)}\right)^0 + c_{p,q} \sum_{(i,j)\in N_p\times N_q} 1_{i,j},$$

where summations over $i_1,\dots,i_{r+1}$ and $n_1,\dots,n_r$ run over some finite sets of natural numbers, with $n_1+\dots+n_r \leqslant \deg(w)$, since the remaining terms will give zero contribution to the considered moment if the product of variables standing to the right of this polynomial is in the matricially free kernel form with respect to $(\varphi_{i,j})$, which is the case if we carry out our computations going from the right to the left.

Note that if $p \neq q$, then $s = 1$, but if $p = q$, then $s \leqslant \deg(w)$. Let us also point out that thanks to our assumption that the array $(X_{i,j})$ has block-identical distributions with respect to $(\varphi_{i,j})$, the same constant $c_{p,q}$ stands by each $1_{i,j}$ for any $i \in N_p$, $j \in N_q$, i.e. it does not depend on i, j. Moreover,

$$\psi_{p,q}\Bigg(\sum_{(i,j)\in N_p\times N_q} 1_{i,j}\Bigg) = \frac{1}{n_q}\sum_{k\in N_q}\varphi_k\Bigg(\sum_{(i,j)\in N_p\times N_q} 1_{i,j}\Bigg) = \frac{1}{n_q}\sum_{(i,j)\in N_p\times N_q}\varphi_j(1_{i,j}) = n_p$$

for any p, q, whereas the first sum in the above expression for $w(S_{p,q})$ belongs to $\operatorname{Ker}(\psi_{p,q})$. These arguments lead us to the conclusion that in the computations of mixed moments of the considered type we can take each polynomial w_k to be of the form $w(S_{p_k,q_k})$ given above, with $c_{p_k,q_k} = 0$ for each $k \in [m]$ since the product $w_1w_2\dots w_m$ is assumed to be in the matricially free kernel form with respect to $(\psi_{p,q})$.

Therefore, each moment of type $\varphi_j(w\mathbf{1}_{p,q}w_1w_2\dots w_m)$ is a sum of moments of type $\varphi_j(w\mathbf{1}_{p,q}v_1v_2\dots v_h)$, where the product $v_1v_2\dots v_h$ is in the matricially free kernel form with respect to $(\varphi_{i,j})$. Similarly, each moment of type $\varphi_j(ww_1w_2\dots w_m)$ is a corresponding sum of moments of type $\varphi_j(wv_1v_2\dots v_h)$ since the reduction described above does not depend on what stands before the product $w_1w_2\dots w_m$. It remains to observe that under each φ_j, the block unit $\mathbf{1}_{p,q}$ acts as the projection onto the linear span of products $v_1v_2\dots v_h$ which are in the matricially free kernel form with respect to $(\varphi_{i,j})$ and begin with $v_1 \in \mathcal{A}_{i_1,j_1}$, where $i_1 \in N_q$. The proof of that fact follows from the definition of $\mathbf{1}_{p,q}$ expressing it in terms of units $1_{i,k}$ which, under φ_j, act as projections onto the linear span of $v_1 v_2 \dots v_h$ which are in the matricially free kernel form with respect to $(\varphi_{i,j})$ and begin with $v_1 \in \mathcal{A}_{i_1,j_1}$, where $i_1 = k$. Of course, we use here the fact that $(p,q) \neq (p_1,q_1)$. This completes the proof of condition (2) of Definition 3.1.

Moreover, using the normalization conditions for $1_{i,j}$'s, we obtain the normalization conditions

$$\psi_r(\mathbf{1}_{p,q}) = \frac{1}{n_r} \sum_{j \in N_r} \varphi_j(\mathbf{1}_{p,q}) = \delta_{r,q},$$

which completes the proof that $(\mathbf{1}_{p,q})$ is a matricially free array of units.

Finally, the proof of condition (1) of Definition 3.2 is similar to that of condition (2) of Definition 3.1 presented above and is based on reducing the computations of the mixed moments $\varphi_j(w_1 w_2 \dots w_n)$, where $w_1 w_2 \dots w_n$ is in the matricially free kernel form with respect to $(\psi_{p,q})$, to mixed moments of type $\varphi_j(v_1 v_2 \dots v_h)$, where $v_1 v_2 \dots v_h$ is in the matricially free kernel form with respect to $(\varphi_{i,j})$. This completes the proof. $\square$

7. Asymptotic matricial freeness of blocks

In this section we study the asymptotic joint distributions of blocks of random pseudomatrices. In particular, we show that they are 'asymptotically matricially free', which is a notion analogous to asymptotic freeness. This generalizes the results of Section 6, where we proved matricial freeness of blocks in the case when the array of matricially free variables has block-identical distributions.

For that purpose we will use a realization on the Fock space $\mathcal{F}$. Having established the realization of the limit laws of random pseudomatrices $S(n)$ under normalized partial traces in Lemma 5.1, it is natural to expect that it can be carried over to the level of *blocks* of $S(n)$ of the form

$$S_{p,q}(n) = \sum_{(i,j) \in N_p \times N_q} X_{i,j}(n)$$

for any n, where we require that the arrays $(X_{i,j}(n))$ of self-adjoint random variables satisfy the assumptions of Theorem 3.1.

Before we proceed with examining the limit joint distribution of these blocks, we define the notion of asymptotic matricial freeness, following the analogous notion of asymptotic freeness. By the *block units* we shall understand units $\mathbf{1}_{p,q}(n)$ defined for each n in terms of $1_{i,j}(n)$'s in exactly the same way as in Section 6. In addition to the states $\phi(n)$ and $\psi(n)$ on the algebras $\mathcal{A}(n)$, we will also use normalized partial traces $\psi_q(n)$, where $q \in [r]$ and $n \in \mathbb{N}$, defined as before in the case of fixed n. Informally, the states $\psi_q(n)$ will play the role of conjugate states which converge to the states Ψ_q of Section 4 as $n \to \infty$.

Definition 7.1. Let $(\phi_{p,q}(n))$ be an r-dimensional array of functionals on the algebra of noncommutative polynomials $\mathbb{C}\langle S_{p,q}(n), \mathbf{1}_{p,q}(n), p,q \in [r]\rangle$. We will say that $(S_{p,q}(n))$ is *asymptotically matricially free* with respect to $(\phi_{p,q}(n))$ if these functionals have pointwise limits $(\phi_{p,q})$ as $n \to \infty$ with respect to which the limit array is matricially free.

We shall say that the tuple $((i_1, j_1), \ldots, (i_m, j_m))$ *defines* a partition $\pi = \{\pi_1, \ldots, \pi_r\}$ of $[m]$ if the equation $(i_k, j_k) = (i_l, j_l)$ holds if and only if k and l belong to the same block of π. In this context it should be observed that if $\pi \in \mathcal{NC}^2_m((i_1, j_1), \ldots, (i_m, j_m))$, then it does not mean that $((i_1, j_1), \ldots, (i_m, j_m))$ defines π (the converse implication does not hold, either). Moreover, if $\pi \in \mathcal{NC}^2_m((p_1, q_1), \ldots, (p_m, q_m))$, where $p_1, q_1, \ldots, p_m, q_m$ label the blocks associated with the decomposition $[n] = N_1 \cup \cdots \cup N_r$, it does not mean that $\pi \in \mathcal{NC}^2_m((i_1, j_1), \ldots, (i_m, j_m))$ for any $(i_k, j_k) \in N_{p_k} \times N_{q_k}$, where $k \in [m]$.

Theorem 7.1. *Under the assumptions of Theorem* 3.1*, the joint $\psi_q(n)$-distributions of blocks and block units converge to the joint Ψ_q-distributions of the truncated matricially free Gaussian operators and the truncated units, respectively, as $n \to \infty$.*

Proof. This result is a refinement of Theorem 3.1. The combinatorial arguments referring to non-crossing partitions used in [5, Lemma 6.1] can be repeated, except that $\psi(n)$ is replaced by $\psi_q(n)$ and we take products of blocks $S_{p,q}(n)$ instead of powers of random pseudomatrices $S(n)$. Clearly, it suffices to consider the case of m even, say $m = 2s$, since if m is odd, both sides are zero by standard arguments. We will first show that

$$\lim_{n\to\infty} \psi_q(n)\big(S_{p_1,q_1}(n)\ldots S_{p_m,q_m}(n)\big) = \Psi_q(\omega_{p_1,q_1}\ldots\omega_{p_m,q_m})$$

for any $q \neq 0$. For given q and $p_1, q_1, \ldots, p_m, q_m$, in order to compute the left-hand side of the above equation, it suffices to take into account the mixed moments

$$\phi_j(n)\big(X_{i_1,j_1}(n)\ldots X_{i_m,j_m}(n)\big)$$

in which $(i_k, j_k) \in N_{p_k} \times N_{q_k}$ for any $k \in [m]$ and $j \in N_q$.

However, in the limit $n \to \infty$, further reductions take place. As in the proof for random pseudomatrices [5, Lemma 6.1], the sum of mixed moments of matricially free random variables in the states $\phi_j(n)$ of the above type whose tuples of indices $((i_1, j_1), \ldots, (i_m, j_m))$ define partitions π of the set $[m]$ which are not non-crossing pair partitions is $O(1/\sqrt{n})$.

Moreover, it suffices to take into account the mixed moments of the above type whose indices define $\pi \in \mathcal{NC}^2_{m,j}(i_1, j_1, \ldots, i_m, j_m)$. In other words, it is enough to consider the moments which are compatible with the matricial multiplication of $(X_{i,j}(n))$ and the vectors onto which their product acts. In particular, this implies that $\pi \in \mathcal{NC}^2_{m,q}(p_1, q_1, \ldots, p_m, q_m)$ and we can observe that for each partition from this set there exist tuples which satisfy the former (stronger) condition.

In fact, we will show that if the moment of the above type defines a non-crossing pair partition π, then the following implication holds:

$$\phi_j(n)\big(X_{i_1,j_1}(n)\ldots X_{i_m,j_m}(n)\big) \neq 0 \quad\Longrightarrow\quad \pi \in \mathcal{NC}^2_{m,j}(i_1, j_1, \ldots, i_m, j_m).$$

In order to show that, we will use an inductive argument and assume that this implication holds for moments of even orders $\leqslant m-2$. Suppose that the moment is not equal to zero and that $\{k, k+1\}$ is the last block which does not have inner blocks. Then the numbers which follow, namely $k+2, \ldots, m$, must belong to $\mathcal{R}(\pi)$ and $(i_{k+2}, j_{k+2}) \neq \cdots \neq (i_m, j_m)$. Therefore, the above moment must be equal to

$$\phi_j(n)\big(X_{i_1,j_1}(n)\dots X_{i_{k-1},j_{k-1}}(n)1_{i_k,j_k}X_{i_{k+2},j_{k+2}}(n)\dots X_{i_m,j_m}(n)\big)$$

multiplied by $\phi_{j_k}(n)(X^2_{i_k,j_k}(n)) = u_{p,q}/n$ for some p,q, where we used assumptions (A1)–(A3) of Section 3 and the fact that the remaining term contains

$$X^2_{i_k,j_k}(n) - \phi_{j_k}(n)\big(X^2_{i_k,j_k}(n)\big)1_{i_k,j_k} \in \operatorname{Ker}\phi_{j_k}(n),$$

and this term must vanish by [5, Lemma 4.1] since we supposed that π defines a pair partition. Since each variance $u_{p,q}$ is positive, the above moment with the unit $1_{i_k,j_k}$ in the middle does not vanish, which implies that $1_{i_k,j_k}$ can be deleted on the right-hand side (see Definition 3.1) and thus we must have one of the two cases:

(i) $j_{k-1} = i_{k+2}$, which corresponds to the case when $k-1 \in \mathcal{R}(\pi)$,
(ii) $(i_{k-1}, j_{k-1}) = (i_{k+2}, j_{k+2})$, which corresponds to the case when $k-1 \in \mathcal{L}(\pi)$,

and we must have $j_{k+1} = i_{k+2}, j_{k+2} = i_{k+3}, \dots, j_{m-1} = i_m$ and $j_m = j$, which is a consequence of matricial freeness, see [5, Proposition 2.2]. By the inductive assumption,

$$\pi' \in \mathcal{NC}^2_{m-2,j}(i_1, j_1, \dots, i_{k-1}, j_{k-1}, i_{k+2}, j_{k+2}, \dots i_m, j_m),$$

where π' is the partition obtained from π by removing the block $\{k, k+1\}$. However, it can also be seen that in both cases adding the block $\{k, k+1\}$ to π' gives a partition $\pi \in \mathcal{NC}^2_{m,j}(i_1, j_1, \dots, i_m, j_m)$.

Now, if the indices define $\pi \in \mathcal{NC}^2_{m,j}(i_1, j_1, \dots, i_m, j_m)$, then pulling out the variance corresponding to the block which has no inner blocks and repeating this s times allows us to express each such moment as

$$\phi_j(n)\big(X_{i_1,j_1}(n)\dots X_{i_m,j_m}(n)\big) = \frac{u_q(\pi, f)}{n^s},$$

where we used our multiplicative notation of Definition 2.1 and where f is the unique coloring of π defined by indices p_k associated with $k \in \mathcal{L}(\pi)$, with $q = q_m$ coloring the imaginary block. Since the coloring f is uniquely determined by $p_1, q_1, \dots, p_m, q_m$, we will use the simplified notation $u^*_q(\pi) = u_q(\pi, f)$.

The cardinality of the set of all tuples $(i_1, j_1, \dots, i_m, j_m)$ which define a given partition $\pi \in \mathcal{NC}^2_{m,j}(i_1, j_1, \dots, i_m, j_m)$ is then determined by the number of independent indices $i_{l(1)}, \dots, i_{l(s)}$ associated with the left legs of π, namely

$$\mathcal{L}(\pi) = \big\{l(1), l(2), \dots, l(s)\big\},$$

and by the index j associated with the imaginary block. In turn, these indices are associated with block-indices $p_{l(1)}, \dots, p_{l(s)}$ and $q = q_m$, respectively, and the cardinality of our class is of the same order as

$$\Theta_n(\pi) = n_{q_m} n_{p_{l(1)}} \dots n_{p_{l(s)}} = O\big(n^{s+1}\big)$$

as $n \to \infty$. In fact, as Example 7.1 shows, the exact cardinality is usually slightly smaller than $\Theta_n(\pi)$ since all independent indices i_l, where $l \in \mathcal{L}(\pi)$, must be different in this computation to give a non-zero contribution to the limit. In the formula for $\Theta_n(\pi)$ given above this is not the case since the mapping $\mathcal{L}(\pi) \to [r]$ given by $k \to p(k)$ may not be injective. Nevertheless, we can substitute $\Theta_n(\pi)$ for the cardinality of the considered set when taking the limit $n \to \infty$ since the difference between these two cardinalities is $O(n^s)$.

Therefore, from each considered partition we obtain the contribution

$$\lim_{n\to\infty} \frac{u_q^*(\pi)\Theta_n(\pi)}{n_q n^s} = u_q^*(\pi) d_{p_{l(1)}} \ldots d_{p_{l(s)}},$$

where the division by n_q comes from the normalization of the partial trace $\psi_q(n)$.

Collecting contributions associated with all $\pi \in \mathcal{NC}^2_{m,q}(p_1, q_1, \ldots, p_m, q_m)$, we obtain

$$\lim_{n\to\infty} \psi_q(n)\big(S_{p_1,q_1}(n) \ldots S_{p_m,q_m}(n)\big) = \sum_{\pi \in \mathcal{NC}^2_{m,q}(p_1,q_1,\ldots,p_m,q_m)} b_q^*(\pi),$$

where $b_q^*(\pi) = b_q(\pi, f)$ corresponds to the unique coloring f of π defined by the tuple $(p_1, q_1, \ldots, p_m, q_m)$ and to the matrix $B = DU$. Finally, the proof that the expression on the right-hand side is equal to $\Psi_q(\omega_{p_1,q_1} \ldots \omega_{p_m,q_m})$ is similar to that of Lemma 5.1, which completes the proof. □

Example 7.1. Consider the partition π in Fig. 1 and suppose that $[n] = N_1 \cup N_2$ is a decomposition of $[n]$ into two disjoint intervals for large n and let $n_1 = |N_1|$, $n_2 = |N_2|$. If $(p_1, q_1) = (p_6, q_6) = (1, 2)$, $(p_2, q_2) = (p_3, q_3) = (2, 1)$ and $(p_4, q_4) = (p_5, q_5) = (1, 1)$, then $\pi \in \mathcal{NC}^2_6(p_1, q_1, \ldots, p_6, q_6)$ according to Definition 2.2. The associated non-vanishing mixed moments of $(X_{i,j}(n))$ are of the form

$$\phi_j(n)\big(X_{i,j}(n)X^2_{k,i}(n)X^2_{r,i}(n)X_{i,j}(n)\big) = \frac{u_{1,2}u_{2,1}u_{1,1}}{n^3},$$

where $i, r \in N_1$ and $j, k \in N_2$. In particular, if all indices i, j, k, r are different, then they define $\pi \in \mathcal{NC}^2_{m,j}(i_1, j_1, \ldots, i_6, j_6)$. Moreover, the cardinality of the corresponding set of tuples $(i_1, j_1, \ldots, i_6, j_6)$ is $c = n_1 n_2(n_1 - 1)(n_2 - 1)$. The contribution from such mixed moments survives in the limit $n \to \infty$ and gives $u_{1,2}u_{2,1}u_{1,1}$. There are three other cases: (i) if $i = r$ and $j = k$, then $c = n_1 n_2$, (ii) if $i = r$ and $j \neq k$, then $c = n_1 n_2(n_2 - 1)$, (iii) if $i \neq r$ and $j = k$, then $c = n_1 n_2(n_1 - 1)$. After dividing by n^4, the contribution from such moments is $O(1/n)$.

Corollary 7.1. *Under the assumptions of Theorem* 7.1*, the joint $\psi(n)$-distributions of blocks and block units converge to the joint Ψ-distribution of the truncated matricially free Gaussian operators and truncated units, respectively, as $n \to \infty$.*

Proof. If we replace $\psi_q(n)$ by $\psi(n)$ at the end of the proof of Theorem 7.1, we need to sum over $q \in [r]$ the corresponding right-hand sides with weights d_q, respectively, which result from normalizations of partial traces. Therefore, in order to obtain a combinatorial expression for the corresponding mixed moment, it suffices to replace $b_q^*(\pi)$ in the above formula by $b^*(\pi) = \sum_q d_q b_q^*(\pi)$. This gives $\Psi(\omega_{p_1,q_1} \ldots \omega_{p_m,q_m})$, which proves our assertion. □

Corollary 7.2. *Under the assumptions of Theorem* 7.1*, the array* $(S_{p,q}(n))$ *is asymptotically matricially free with respect to the array of states* $(\psi_{p,q}(n))$ *on* $\mathcal{A}(n)$ *defined by the family of normalized partial traces* $(\psi_q(n))_{1\leqslant q\leqslant r}$ *as* $n\to\infty$.

Proof. It follows from Theorem 7.1 that the $\psi_q(n)$-distribution of $(S_{p,q}(n))$ converges to the Ψ_q-distribution of $(\omega_{p,q})$ for any p,q. Thus, Theorem 7.1 and Proposition 4.2 give our assertion. $\square$

Theorem 7.2. *Under assumptions* (A1)–(A4)*, the joint* $\phi(n)$*-distributions of blocks and block units converge to the joint* Φ*-distribution of the matricially free Gaussian operators and the associated units, respectively, as* $n\to\infty$.

Proof. The proof is similar to that of Theorem 7.1 and is based on the combinatorial arguments of type used in [5, Lemma 6.1] which lead to [5, Lemma 6.2]. $\square$

Corollary 7.3. *Under assumptions* (A1)–(A4)*, the array* $(S_{p,q}(n))$ *is asymptotically matricially free with respect to the array* $(\varphi_{p,q}(n))$ *of states on* $\mathcal{A}(n)$ *defined by* $\phi(n)$ *and the family of normalized partial traces* $(\psi_q(n))_{1\leqslant q\leqslant r}$ *as* $n\to\infty$.

Proof. The assertion follows from Theorem 7.2 and Proposition 4.1. $\square$

8. Symmetric matricial freeness

In order to compare the asymptotics of blocks of random pseudomatrices with that of symmetric random blocks, we shall now introduce a symmetric analogue of matricial freeness.

Roughly speaking, in this concept one replaces ordered pairs by two-element sets and assumes that the array $(\mathcal{A}_{i,j})$ of subalgebras of a unital algebra $\mathcal{A}$ contains the diagonal and is symmetric. Moreover, each algebra $\mathcal{A}_{i,j}$ contains an internal unit $1_{i,j}$ which agrees with $1_{j,i}$ for any $(i,j)\in J$. By $\mathcal{I}$ we denote the unital algebra generated by the internal units and we assume that it is commutative. By $(\varphi_{i,j})$ we denote an array of states on $\mathcal{A}$.

Instead of sets Λ_m, we shall use their symmetric counterparts, namely subsets of I^m of the form

$$\Pi_m=\bigl\{\bigl(\{i_1,i_2\},\{i_2,i_3\},\ldots,\{i_m,i_{m+1}\}\bigr)\colon\ \{i_1,i_2\}\neq\{i_2,i_3\}\neq\cdots\neq\{i_m,i_{m+1}\}\bigr\},$$

where $m\in\mathbb{N}$, with their union denoted $\Pi=\bigcup_{m=1}^{\infty}\Pi_m$. The main difference between 'symmetric matricial freeness' and matricial freeness is that in all definitions we have to use Π instead of Λ.

Definition 8.1. We say that the array $(1_{i,j})$ is a *symmetrically matricially free array of units* associated with $(\mathcal{A}_{i,j})$ and $(\varphi_{i,j})$ if for any diagonal state φ it holds that

1. $\varphi(u_1au_2)=\varphi(u_1)\varphi(a)\varphi(u_2)$ for any $a\in\mathcal{A}$ and $u_1,u_2\in\mathcal{I}$,
2. if $a_k\in\mathcal{A}_{i_k,j_k}\cap\operatorname{Ker}\varphi_{i_k,j_k}$, where $1<k\leqslant m$, then

$$\varphi(a1_{i_1,j_1}a_2\ldots a_m)=\begin{cases}\varphi(aa_2\ldots a_m) & \text{if } (\{i_1,j_1\},\ldots,\{i_m,j_m\})\in\Pi\\ 0 & \text{otherwise,}\end{cases}$$

where $a\in\mathcal{A}$ is arbitrary and $\{i_1,j_1\}\neq\cdots\neq\{i_m,j_m\}$.

Definition 8.2. We say that a symmetric array $(\mathcal{A}_{i,j})$ is *symmetrically matricially free* with respect to $(\varphi_{i,j})$ if

1. for any $a_k \in \operatorname{Ker}\varphi_{i_k,j_k} \cap \mathcal{A}_{i_k,j_k}$, where $k \in [m]$ and $\{i_1, j_1\} \neq \cdots \neq \{i_m, j_m\}$, and for any diagonal state φ it holds that

$$\varphi(a_1 a_2 \ldots a_m) = 0,$$

2. $(1_{i,j})$ is a symmetrically matricially free array of units associated with $(\mathcal{A}_{i,j})$ and $(\varphi_{i,j})$.

The array of variables $(a_{i,j})$ in a unital algebra $\mathcal{A}$ will be called *symmetrically matricially free* with respect to $(\varphi_{i,j})$ if there exists a symmetrically matricially free array of units $(1_{i,j})$ in $\mathcal{A}$ such that the array of algebras $(\mathcal{A}_{i,j})$, each generated by $a_{i,j} + a_{j,i}$ and $1_{i,j}$, respectively, is symmetrically matricially free with respect to $(\varphi_{i,j})$. The definition of $*$-symmetrically matricially free arrays of variables is similar to that of $*$-matricially free arrays.

We shall need the symmetrized operators on the matricially free-boolean Fock space $\mathcal{F}$ of Section 4, like the $\hat{\omega}_{i,j}$'s defined as

$$\hat{\omega}_{i,j} = \begin{cases} \omega_{j,j} & \text{if } i = j \\ \omega_{i,j} + \omega_{j,i} & \text{if } i \neq j. \end{cases}$$

In a similar way we define $\hat{\varsigma}_{i,j}$, $\hat{\zeta}_{i,j}$, $\hat{\wp}_{i,j}$, etc. All these arrays are associated with matrix $A = (\alpha_{i,j})$, which is suppressed in the notation. In turn, the symmetrized units on $\mathcal{F}$ and $\mathcal{F} \ominus \mathbb{C}\Omega$, are of the form

$$\hat{1}_{i,j} = 1_{i,j} + 1_{j,i} - 1_{i,j}1_{j,i} \quad \text{and} \quad \hat{t}_{i,j} = \hat{1}_{i,j}P,$$

respectively, for any i, j, where $(1_{i,j})$ is the array of canonical units on $\mathcal{F}$ and P stands for the projection onto $\mathcal{F} \ominus \mathbb{C}\Omega$. More explicitly, $\hat{1}_{i,j}$ is the projection onto the subspace of $\mathcal{F}$ spanned by tensors which begin with $e_{i,k}$ or $e_{j,k}$ for some k, and, in addition, by Ω if $i = j$.

Proposition 8.1. *If the matrix $A = (\alpha_{i,j})$ is symmetric, then*

1. *the Ψ_j-distribution of $\hat{\omega}_{i,j}$ is the semicircle law of radius $2\alpha_{i,j}$ for any (i, j),*
2. *the array $(\hat{\omega}_{i,j})$ is symmetrically matricially free with respect to $(\Psi_{i,j})$.*

Proof. If $i = j$, then the first claim is similar to (1) of Proposition 4.1 since the action of $\omega_{j,j}$ onto $e_{j,j}$ is similar to the action of $\zeta_{j,j}$ onto the vacuum vector. If $i \neq j$, then for even and positive m, it holds that

$$\Psi_j\big(\hat{\omega}_{i,j}^m\big) = \sum_{\epsilon_1,\ldots,\epsilon_m \in \{1,*\}} \; \sum_{(i_1,j_1),\ldots,(i_m,j_m) \in \{(i,j),(j,i)\}} \Psi_j\big(\ell^{\epsilon_1}(e_{i_1,j_1}) \ldots \ell^{\epsilon_m}(e_{i_m,j_m})\big)$$

and there is a bijection between $\mathcal{NC}_m^2$ and products of the form $\ell^{\epsilon_1}(e_{i_1,j_1}) \ldots \ell^{\epsilon_m}(e_{i_m,j_m})$ whose action onto $e_{j,j}$ is non-trivial. This bijection is obtained as follows. If $\pi \in \mathcal{NC}_m^2$ is given and $\{l, r\}$ is a block of π, where $l < r$, then $\epsilon_l = *$, $\epsilon_r = 1$ and $(i_l, j_l) = (i_r, j_r)$ with

$$(i_r, j_r) = \begin{cases} (j_{o(r)}, i_{o(r)}) & \text{if } \{l, r\} \text{ has an outer block} \\ (i, j) & \text{otherwise,} \end{cases}$$

where $o(r)$ is the right leg of the nearest outer block of $\{l, r\}$. It can be seen that this mapping is onto since in order to get a non-trivial action on $e_{j,j}$ of the considered product of operators, the action of each $\ell(e_{i,j})$ (corresponding to the right leg of some block) must be followed by the action of $\ell(e_{j,i})$ (corresponding to the right leg of another block) or $\ell^*(e_{i,j})$ (corresponding to the left leg of the same block). In particular, acting first with $\ell(e_{i,j})$ on a vector from $\mathcal{F}$ and then with $\ell(e_{i,j})$ or $\ell^*(e_{j,i})$ gives zero. This completes the proof of (1) since the even moments of the semicircle law of radius $2\alpha_{i,j}$ are given by $M_m = \alpha_{i,j}^m c_{m/2}$, where $c_{m/2} = |\mathcal{NC}_m^2|$, $m \in 2\mathbb{N}$, are Catalan numbers. The proof of (2) is similar to that of (2) of Proposition 4.1. We can show the slightly more general result that the array $\hat{\mathcal{B}}_{i,j} = \mathbb{C}\langle \hat{\wp}_{i,j}, \hat{\wp}_{i,j}^*, \hat{t}_{i,j} \rangle$ is symmetrically matricially free with respect to $(\Psi_{i,j})$, where we use relations

$$\hat{\wp}_{i,j}^* \hat{\wp}_{i,j} = b_{i,j} \hat{t}_{i,j},$$

and normalization $\Psi_j(\hat{t}_{i,j}) = 1$ for any $(i, j) \in J$. The details are left to the reader. □

For a given partition $[n] = N_1 \cup N_2 \cup \cdots \cup N_r$ of the form considered before, it is useful to introduce the notation

$$N_{p,q} = (N_p \times N_q) \cup (N_q \times N_p)$$

for sets of pairs which label 'symmetric blocks' of random pseudomatrices. If we let $n \to \infty$ and assume that blocks grow as before, we can obtain the asymptotic behavior of these symmetric blocks.

Theorem 8.1. *Under the assumptions of Theorem* 7.1*, the array of symmetric blocks given by*

$$Z_{p,q}(n) := \sum_{(i,j) \in N_{p,q}} X_{i,j}(n)$$

is asymptotically symmetrically matricially free with respect to $(\psi_{p,q}(n))$ *as* $n \to \infty$.

Proof. Observing that

$$Z_{p,q}(n) = \begin{cases} S_{p,p}(n) & \text{if } p = q \\ S_{p,q}(n) + S_{q,p}(n) & \text{otherwise,} \end{cases}$$

it suffices to use Theorem 7.1 and Proposition 8.1 to prove the assertion. □

9. Symmetric random blocks

We show in this section that the tracial asymptotics of random pseudomatrices is the same as the asymptotics of complex Gaussian random matrices and that this similarity can be carried over to the level of their symmetric random blocks.

The context for the study of random matrices originated by Voiculescu [9] is the following. Let μ be a probability measure on some measurable space without atoms and let $L = \bigcap_{1\leqslant p<\infty} L^p(\mu)$ be endowed with the state expectation $\mathbb{E}$ given by integration with respect to μ. The $*$-algebra of $n \times n$ random matrices is $M_n(L) = L \otimes M_n(\mathbb{C})$ with the state $\tau(n) = \mathbb{E} \otimes \mathrm{tr}(n)$, where $\mathrm{tr}(n)$ is the normalized trace.

In order to compare the asymptotics of the blocks of random matrices with that of random pseudomatrices, we partition each set $[n]$ into disjoint non-empty intervals as in the case of pseudomatrices and we set again $D = \mathrm{diag}(d_1, d_2, \dots, d_r)$ to be the associated diagonal dimension matrix.

By a *complex Gaussian random matrix* we understand a matrix $(Y_{i,j}(n))_{1\leqslant i,j\leqslant n}$, in which $Y_{i,j}(n) = \overline{Y_{j,i}(n)}$ for any i, j, n and

$$\bigl\{\mathrm{Re}\, Y_{i,j}(n) \bigm| 1 \leqslant i \leqslant j \leqslant n\bigr\} \cup \bigl\{\mathrm{Im}\, Y_{i,j}(n) \bigm| 1 \leqslant i \leqslant j \leqslant n\bigr\}$$

is an independent set of Gaussian random variables. Its submatrices of the form

$$T_{p,q}(n) = \sum_{(i,j)\in N_{p,q}} Y_{i,j}(n) \otimes e_{i,j}(n)$$

will be called *symmetric random blocks*, where $p, q \in [r]$ and $\{e_{i,j}(n) \mid 1 \leqslant i \leqslant j \leqslant n\}$ is a system of matrix units.

We will study the asymptotics of symmetric random blocks under some natural assumptions. Our setting is very similar to that in [9,12] except that we will assume that the variances of $|Y_{i,j}(n)|$, where $i, j \in [n]$ and n is fixed, are block-identical rather than identical. The case of non-Gaussian random matrices [2] and the corresponding symmetric blocks can be treated in a similar way.

In order to find a Hilbert space realization of the limit joint distribution of symmetric random blocks under $\tau(n)$, we will use the symmetrized truncated Gaussian operators $\hat{\omega}_{p,q}$ on $\mathcal{F}$ and the state $\Psi = \sum_j d_j \Psi_j$ on $B(\mathcal{F})$, where the dependence of $\hat{\omega}_{p,q}$'s on the matrix $B = DU$ is suppressed in the notation.

Theorem 9.1. *Let $(Y_{i,j}(n))_{1\leqslant i,j\leqslant n}$ be a complex Gaussian random matrix for each $n \in \mathbb{N}$ such that*

1. $\mathbb{E}(Y_{i,j}(n)) = 0$ *for any i, j, n,*
2. $\mathbb{E}(|Y_{i,j}(n)|^2) = u_{p,q}/n$ *for any $i \in N_p$ and $j \in N_q$ and any n,*

where $U := (u_{p,q}) \in M_r(\mathbb{R})$. Then it holds that

$$\lim_{n\to\infty} \tau(n)\bigl(T_{p_1,q_1}(n) \dots T_{p_m,q_m}(n)\bigr) = \Psi(\hat{\omega}_{p_1,q_1} \dots \hat{\omega}_{p_m,q_m})$$

for any $p_1, q_1, \dots, p_m, q_m \in [r]$, where $\hat{\omega}_{p,q}$'s are associated with $B := DU$, where D is the dimension matrix.

Proof. Our proof refers to the original proof of Voiculescu [9,12], which is followed by some combinatorial arguments referring to non-crossing pair partitions (in our approach the variances may vary). Let $\tau_q(n) = \mathbb{E} \otimes \mathrm{tr}_q$, where $\mathrm{tr}_q(A) = 1/n_q \sum_{j \in N_q} A_{j,j}$ is the normalized partial trace of $A \in M_n(\mathbb{C})$ corresponding to $q \in [r]$. We have

$$\tau_q(n)\big(T_{p_1,q_1}(n) \dots T_{p_m,q_m}(n)\big)$$
$$= \sum_{(i_1,j_1)\in N_{p_1,q_1},\dots,(i_m,j_m)\in N_{p_m,q_m}} \mathbb{E}\big(Y_{i_1,j_1}(n) \dots Y_{i_m,j_m}(n)\big) \mathrm{tr}_q(e_{i_1,j_1} \dots e_{i_m,j_m}).$$

Since all terms are zero for m odd, throughout the rest of the proof we assume that $m = 2s$ for some $s \in \mathbb{N}$.

An individual term of this sum is then non-zero only if

$$j_1 = i_2, \qquad j_2 = i_3, \quad \dots, \quad j_m = i_1 \in N_q$$

and there exists a bijection $\gamma : [m] \to [m]$ such that

$$\gamma^2 = \mathrm{id}, \qquad \gamma(k) \neq k \quad \text{and} \quad i_{\gamma(k)} = j_k, \qquad j_{\gamma(k)} = i_k.$$

Each such term is $O(n^{-s-1})$ as $n \to \infty$. For convenience, we restrict our attention to the case when

$$(i_1, j_1) \in N_{p_1} \times N_{q_1}, \quad \dots, \quad (i_m, j_m) \in N_{p_m} \times N_{q_m},$$

since the remaining cases can be treated in a similar way, with some p_i's interchanged with the corresponding q_i's.

The number of non-zero terms of the considered type is $\sum_\gamma \Theta_n(\gamma)$, where γ runs over the set of permutations of $[m]$ such that

$$\gamma^2 = \mathrm{id}, \qquad \gamma(k) \neq k \quad \text{and} \quad p_k = q_{\gamma(k)}, \qquad q_k = p_{\gamma(k)},$$

for any $k \in [m]$. If $q = p_1$, we obtain

$$\Theta_n(\gamma) = \mathrm{card}\big\{(i_1, \dots, i_m) \in N_{p_1} \times \dots \times N_{p_m}\colon\ i_k = i_{\gamma(k)+1},\ i_{k+1} = i_{\gamma(k)}\big\},$$

where addition is modulo m. Denote by $\pi(\gamma)$ the pair-partition of $[m]$ defined by γ.

If $\pi(\gamma)$ is a crossing pair-partition, then $\Theta_n(\gamma) \leqslant O(n^s)$, which can be justified as follows. Let $k < l < \gamma(k) < \gamma(l)$ be a quadruple corresponding to a crossing between blocks $\{k, \gamma(k)\}$ and $\{l, \gamma(l)\}$. Then, the conditions on $i_1, \dots, i_m$ given above lead to the equation $i_{\gamma(k)} = i_{\gamma(l)}$ since we must have $i_{\gamma(k)} = j_l$ (by matricial multiplication) as well as $j_l = i_{\gamma(l)}$. Apart from this equation, we have conditions on $i_1, \dots, i_m$ obtained from matricial multiplication related to those neighboring blocks which do not have crossings (these reduce the number of independent indices to $s + 1$ associated with $\mathcal{R}(\pi)$ and one index associated with the covering block as we demonstrate below when discussing non-crossing pair-partitions). In other words, a crossing further reduces the number of independent indices. Therefore, when we add all mixed moments associated with a crossing pair-partition, the cardinality $\Theta_n(\gamma)$ is at most $O(n^s)$ and thus the

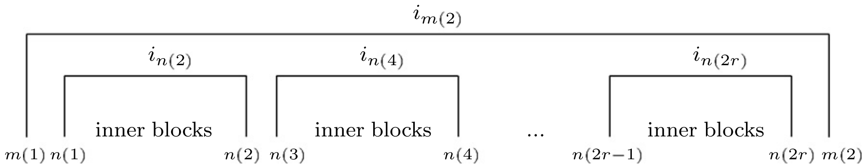


Fig. 2. Inductive step involving subpartitions of $\pi(\gamma)$.

contribution disappears as $n \to \infty$ since we still need to multiply this cardinality by the corresponding expectation, which is $O(1/n^s)$, and divide it by n_q due to the normalization of the partial trace.

In turn, if $\pi(\gamma) \in \mathcal{NC}_m^2 \setminus \mathcal{NC}_{m,q}^2(\{p_1, q_1\}, \ldots, \{p_m, q_m\})$, then the associated moments vanish since they involve products which are not of the type considered above. Let us justify this for the subpartition of $\pi(\gamma)$ shown in Fig. 2, to which the general case can be reduced by induction. If an individual moment associated with this partition does not vanish, then it is already clear that

$$\{p_k, q_k\} = \{p_{\gamma(k)}, q_{\gamma(k)}\}$$

for any $k \in [m]$, which gives condition (a) of Definition 2.3. Now, let us restrict our attention to the outer block shown in Fig. 2 and all blocks of depth one. Using the conditions involving the indices (i_k, j_k) given earlier, we arrive at the equations

$$i_{n(1)} = i_{n(3)} = \cdots = i_{n(2r-1)} = j_{n(2r)} = i_{m(2)},$$

which implies that

$$\{p_{m(2)}, q_{m(2)}\} \cap \bigcap_{k=1}^{2r} \{p_{n(k)}, q_{n(k)}\} \neq \emptyset,$$

and that is condition (b). To show that condition (c) also holds, consider an inner–outer triple (π_k, π_l, π_n) of $\pi(\gamma)$, where $\pi_k = \{k, \gamma(k)\}$, $\pi_l = \{l, \gamma(l)\}$, $\pi_n = \{n, \gamma(n)\}$. To the left leg of the middle block we assign the pair of indices (i_l, j_l) and we already know that $i_l = i_{\gamma(n)}$. Thus, since $i_l \in N_{p_l}$ and $i_{\gamma(n)} \in N_{q_n}$, we must have $p_l = q_n$ since the intervals N_p are identical or disjoint. On the other hand, $i_k = i_{\gamma(l)}$ and since $i_k \in N_{p_k}$ and $i_{\gamma(l)} \in N_{q_l}$, we must have $p_k = q_l$. Therefore, $\{p_l, q_l\}$ must be contained in $\{p_k, q_k\} \cup \{p_n, q_n\}$, which gives (c).

Finally, if $\pi(\gamma) \in \mathcal{NC}_{m,q}^2(\{p_1, q_1\}, \ldots, \{p_m, q_m\})$, then the number of independent indices which enter in the computation of $\Theta_n(\gamma)$ is $s+1$. We claim that for such γ we have

$$\Theta_n(\gamma) = \operatorname{card}\{(i_1, i_{r(1)}, \ldots, i_{r(s)}) \in N_{p_1} \times N_{p_{r(1)}} \times \cdots \times N_{p_{r(s)}}\},$$

where

$$\mathcal{R}(\pi(\gamma)) = \{r(1), r(2), \ldots, r(s)\}.$$

In fact, it is not difficult to show that the conditions which define $\Theta_n(\gamma)$ can be reduced to $s+1$ independent indices $i_1, i_{r(1)}, \ldots, i_{r(s)}$ which can be interpreted as independent colors assuming arbitrary values from the corresponding intervals $N_{p_1}, N_{p_{r(1)}}, \ldots, N_{p_{r(s)}}$, respectively, with index i_1 coloring the imaginary block.

The proof of this fact essentially reduces to the inductive step involving subpartitions of $\pi(\gamma)$ of the form shown in Fig. 2, where the covering block can also be interpreted as the imaginary block associated with $\pi(\gamma)$. We want to show that the conditions which define $\Theta_n(\gamma)$ reduce all indices involved here to those associated with the right legs of the blocks of the subpartition shown above. In particular, on the level of blocks of depth one, we have pairings

$$\gamma\big(n(1)\big) = n(2), \qquad \gamma\big(n(3)\big) = n(4), \quad \ldots, \quad \gamma\big(n(2r-1)\big) = n(2r),$$

and, as we have already shown above, the indices associated with the left legs of these blocks are equal to the index associated with the right leg of its covering block. No blocks of depth greater than one lead to new conditions involving $i_{m(2)}$ and for that reason the index $i_{m(2)}$ can be used to color the covering block of the considered subpartition. The same pattern is repeated for blocks of arbitrary depth which are covered by the same block. The same holds for blocks of zero depth, where the role of the covering block in the above reasoning is played by the imaginary block (in that case the imaginary block is colored by i_1).

This argument can be viewed as a proof of the inductive step of our claim that the conditions involving the indices $i_1, i_2, \ldots, i_m$, used to define the numbers $\Theta_n(\gamma)$ for the 'right' γ reduce to $s+1$ independent indices associated with the right legs of the blocks of π. If the covering block is interpreted as the imaginary block, we obtain the starting case of the induction. This proves our formula for $\Theta_n(\gamma)$, from which we easily get

$$\Theta_n(\gamma) = n_{p_1} n_{p_{r(1)}} \ldots n_{p_{r(s)}}.$$

Let us remark here that not all tuples $(i_1, i_{r(1)}, \ldots, i_{r(s)})$ which contribute to $\Theta_n(\gamma)$, but only those which are pairwise different, are actually used in our computation of the asymptotic joint distribution. If any two (or more) of these indices coincide, the corresponding contribution from the associated expectations is $O(n^{-1})$ and therefore becomes irrelevant in the limit.

For those independent indices which are pairwise different and correspond to partitions $\pi(\gamma) \in \mathcal{NC}_m^2(\{p_1, q_1\}, \ldots, \{p_m, q_m\})$, the corresponding expectation of Gaussian random variables $\mathbb{E}(Y_{i_1,j_1}(n) \ldots Y_{i_m,j_m}(n))$ is the product of

$$\mathbb{E}\big(Y_{i_k,j_k}(n) Y_{i_{\gamma(k)},j_{\gamma(k)}}(n)\big) = v_{i_k,j_k}(n),$$

where $\gamma(k) \in \mathcal{R}(\pi(\gamma))$, with $i_{\gamma(k)} = j_k$ and $j_{\gamma(k)} = i_k$. Now, since $i_k = i_{o(k)}$, the above expectation is equal to $v_{i,j}(n)$, where $i = i_{o(k)}$ and $j = i_{\gamma(k)}$. In other words, j is the color assigned to block $\{k, \gamma(k)\}$ since we have shown before that we color the blocks with indices which correspond to their right legs and i is the color assigned to its covering block. This includes the case of blocks which do not have an outer block, then the role of the latter is played by the imaginary block colored by $i_1 = j_m$.

Now, using our assumption on the variance matrices, $v_{i,j}(n) = v_{j,i}(n) = u_{q,p}/n$ whenever $i \in N_p$, $j \in N_q$. Therefore, taking into account all pairings and using the definition of symbol $u_j(\pi, f)$ of Section 2, we obtain

$$\mathbb{E}\big(Y_{i_1,j_1}(n) \ldots Y_{i_m,j_m}(n)\big) = \frac{u_{p_1}(\pi, f)}{n^s}$$

for the relevant Gaussian expectations of variables associated with $\pi(\gamma)$, where $f \in F_r(\pi(\gamma))$ is the coloring of the associated $\pi(\gamma)$ defined by indices $p_{r(1)}, p_{r(2)}, \ldots, p_{r(s)}$ assigned to the blocks of $\pi(\gamma)$, with color p_1 assigned to the imaginary block.

Collecting expectations corresponding to all $\pi \in \mathcal{NC}_m^2$ and taking into account that

$$\lim_{n\to\infty} \frac{\Theta_n(\gamma)}{n^s} = d_{p_1} d_{p_{r(1)}} \ldots d_{p_{r(s)}},$$

we obtain

$$\lim_{n\to\infty} \tau_q(n)\big(T_{p_1,q_1}(n) \ldots T_{p_m,q_m}(n)\big) = \sum_{\pi \in \mathcal{NC}_m^2(\{p_1,q_1\},\ldots,\{p_m,q_m\})} \hat{b}_q(\pi),$$

where $\hat{b}_q(\pi) = \sum_f b_q(\pi, f)$ is the sum over all admissible colorings of π and each $b_q(\pi, f)$ corresponds to the matrix $B = DU$ and to the color q of the imaginary block.

If we replace $\tau_q(n)$ by $\tau(n)$ in the above expression, we just need to sum over $q \in [r]$ the corresponding right-hand sides with weights d_q, respectively, which result from normalizations of partial traces. Therefore, in order to obtain the required expression for the corresponding mixed moment, we need to replace $\hat{b}_q(\pi)$ by $\hat{b}(\pi) = \sum_q d_q \hat{b}_q(\pi)$, which completes the proof of the combinatorial formula.

It remains to show that the mixed moment of the symmetrized Gaussian operators in the state Ψ gives the appropriate sum of combinatorial expressions obtained above. This, in turn, is similar to the proof of Lemma 5.1. Therefore, our proof is completed. □

Remark 9.1. In this context, let us remark that the array of symmetric random blocks $(T_{p,q}(n))$ is not symmetrically matricially free with respect to $(\tau_{p,q}(n))$ for finite n. In order to see this, it suffices to compute some simple examples of mixed moments of the $T_{p,q}(n)$'s associated with crossing partitions. For instance, if $n = 3$ and the blocks are one-dimensional, then $\tau_1(T_{1,2}T_{2,3}T_{3,1}T_{1,2}T_{2,3}T_{3,1}) \neq 0$, where $\tau_1 = \tau_1(3)$ and $T_{p,q} = T_{p,q}(3)$ for any $1 \leqslant p, q \leqslant 3$, which shows that condition (1) of Definition 8.2 is not satisfied.

However, in view of Proposition 8.1 and Theorem 9.1, we can expect that symmetric random blocks are symmetrically matricially free as $n \to \infty$. The notion of asymptotic symmetric matricial freeness is analogous to that of asymptotic matricial freeness (see Definition 7.1). It remains to define appropriate arrays of block units. Thus, let

$$\hat{\mathbf{1}}_{p,q}(n) = \sum_{j \in N_p \cup N_q} 1 \otimes e_{j,j}(n)$$

for any $p, q \in [r]$ and each $n \in \mathbb{N}$. It is easy to see that $\hat{\mathbf{1}}_{p,q}(n)$ is an internal unit in the algebra generated by $T_{p,q}(n)$ and $\hat{\mathbf{1}}_{p,q}(n)$ for any given $p, q \in [r]$ and $n \in \mathbb{N}$.

Theorem 9.2. *Under the assumptions of Theorem* 9.1*, let $(\tau_{p,q}(n))$ be the r-dimensional array of states on the algebra $\mathcal{A}(n) = L \otimes M_n(\mathbb{C})$ defined by the normalized partial traces $\tau_q(n)$, where $q \in [r]$, for any $n \in \mathbb{N}$. Then the array of symmetric random blocks $(T_{p,q}(n))$ is asymptotically symmetrically matricially free with respect to $(\tau_{p,q}(n))$.*

Proof. In view of Proposition 8.1, it suffices to show that the $\tau_m(n)$-distributions of blocks $T_{p,q}(n)$ and units $\hat{\mathbf{1}}_{p,q}(n)$ converge to the Ψ_m-distribution of the symmetrized Gaussians $\hat{\omega}_{p,q}$ and symmetrized units $\hat{1}_{p,q}$ as $n \to \infty$ for any given m. It suffices to verify that $\hat{\mathbf{1}}_{p,q}(n)$ leaves invariant any vector of the form

$$w_1\big(T_{p_1,q_1}(n)\big)\dots w_k\big(T_{p_k,q_k}(n)\big)e_m,$$

where the product of polynomials $w_1, \dots, w_k$ is in the symmetrically matricially free kernel form with respect to the array $(\tau_{p,q}(n))$ and $\{p,q\} \cap \{p_1,q_1\} \neq \emptyset$, and kills all other vectors. This is quite easy to observe since any vector of this type is a linear combination of the form $\sum_{i\in N_p}\alpha_i a_i \otimes e_i + \sum_{j\in N_q}\beta_j b_j \otimes e_j$, where $a_i, b_j \in L$ and $\alpha_i, \beta_j \in \mathbb{C}$ for any i, j, and thus it is left invariant by $\hat{\mathbf{1}}_{p,q}(n)$, where (e_j) is the canonical basis in $\mathbb{C}^n$. $\square$

10. Asymptotic freeness and asymptotic monotone independence

If we make additional assumptions on the variance matrices $(v_{i,j}(n))$ of matricially free arrays $(X_{i,j}(n))$, we obtain asymptotic freeness which refers to the rows of random pseudomatrices. The proposition given below can also be viewed as an operatorial version of [5, Proposition 8.2].

Proposition 10.1. *If the arrays $(X_{i,j}(n))$ of Theorem 3.2 are square and have identical variances within blocks and rows, then the sums*

$$S_p(n) := \sum_{q=1}^{r} S_{p,q}(n), \quad \text{where } 1 \leqslant p \leqslant r,$$

are asymptotically free with respect to both $\phi(n)$ and $\psi(n)$ as $n \to \infty$.

Proof. In view of Theorems 7.1–7.2, it suffices to show that the variables $\zeta_1, \dots, \zeta_r$ are free with respect to Φ, and that $\omega_1, \dots, \omega_r$ are free with respect to Ψ, where

$$\zeta_p = \sum_{q=1}^{r} \zeta_{p,q} \quad \text{and} \quad \omega_p = \sum_{q=1}^{r} \omega_{p,q},$$

with the notations of Section 4. If $\{e_1, e_2, \dots, e_r\}$ is an orthonormal set of vectors, we have the natural isomorphism

$$\kappa : \mathcal{F} \to \mathcal{F}\bigg(\bigoplus_q \mathbb{C}e_q\bigg)$$

of Remark 4.1 and then, since $\alpha_{p,q} = \alpha_{p,p}$ for any $p, q \in [r]$, we have

$$\zeta_p = \alpha_{p,p}\kappa^*\omega(e_p)\kappa,$$

where $\omega(e_p) = \ell(e_p) + \ell^*(e_p)$ is the canonical free Gaussian operator on $\mathcal{F}(\bigoplus_q \mathbb{C}e_q)$ for any p. Let us remark that ω_p and $\omega(e_p)$ denote different operators in our notation. Since

$\omega(e_1), \ldots, \omega(e_r)$ are free with respect to the vacuum state on $B(\mathcal{F}(\bigoplus_q \mathbb{C}e_q))$, the operators $\zeta_1, \ldots, \zeta_r$ are free with respect to the vacuum state on $B(\mathcal{F})$. Similarly,

$$\omega_p = \alpha_{p,p}\gamma^*\omega(e_p)\gamma$$

for any $p \in [r]$, where $\gamma = P \circ \kappa$ and P is the canonical projection from $\mathcal{F}(\bigoplus_q \mathbb{C}e_q)$ onto the orthocomplement of $\mathbb{C}\Omega$. Now, since $P\omega(e_1)P, \ldots, P\omega(e_r)P$ are free with respect to $\psi_q(.) = \langle .e_q, e_q\rangle$ for any q, the operators $\omega_1, \ldots, \omega_r$ are free with respect to Ψ_q for any q. Moreover, the Ψ_p-distribution of ω_p does not depend on p since the variances are assumed to be identical in each row. Therefore, $\omega_1, \ldots, \omega_r$ are free with respect to Ψ. This completes the proof. $\square$

Analogous results hold for block-triangular arrays and lead to monotone independence (block-lower-triangular arrays) and anti-monotone independence (block-upper-triangular arrays). One has to remember that order is important for these notions of independence and therefore in that case we use sequences of variables instead of families. We formulate the result only for block-lower-triangular arrays since the case of block-upper-triangular arrays is completely analogous. This result can be viewed as an operatorial version of [5, Proposition 8.3].

Proposition 10.2. *If the arrays $(X_{i,j}(n))$ of Theorem* 3.2 *are block-lower-triangular and have identical variances within blocks and rows, then the sums*

$$S_p(n) := \sum_{q=1}^{p} S_{p,q}(n), \quad \text{where } 1 \leqslant p \leqslant r,$$

are asymptotically monotone independent with respect to $\phi(n)$ as $n \to \infty$.

Proof. Let $\zeta_p = \sum_{q=1}^{p} \zeta_{p,q}$, where $p \in [r]$. In view of Theorem 7.1, it suffices to show that the operators $\zeta_1, \ldots, \zeta_r$ are monotone independent with respect to Φ. Let $\mathcal{F}_1$ be the subspace of $\mathcal{F}(\bigoplus_q \mathbb{C}e_q)$ of the form

$$\mathcal{F}_1 = \mathbb{C}\xi \oplus \bigoplus_{m=1}^{r} \bigoplus_{p_1 > \cdots > p_m} \bigoplus_{n_1,\ldots,n_m \in \mathbb{N}} \mathcal{H}_{p_1}^{\otimes n_1} \otimes \cdots \otimes \mathcal{H}_{p_m}^{\otimes n_m}$$

where $\mathcal{H}_q = \mathbb{C}e_q$ for any $q \in [r]$ and $\{e_1, e_2, \ldots, e_r\}$ is an orthonormal basis in some Hilbert space and denote by $Q : \mathcal{F}(\bigoplus_q \mathbb{C}e_q) \to \mathcal{F}_1$ the corresponding canonical projection. Observe that we have

$$\zeta_p = \alpha_{p,p}\beta^*\omega(e_p)\beta$$

where $\beta = Q \circ \kappa$ and κ is the same as in the proof of Proposition 10.1. Since the operators $Q\omega(e_1)Q, \ldots, Q\omega(e_r)Q$ are monotone independent with respect to the vacuum state on the free Fock space, the operators $\zeta_1, \ldots, \zeta_r$ are monotone independent with respect to Φ. This completes the proof. $\square$

Finally, we obtain asymptotic boolean independence of blocks of block-diagonal pseudomatrices.

Proposition 10.3. *If the arrays* $(X_{i,j}(n))$ *of Theorem* 3.2 *are block-diagonal and have identical variances within blocks, then the sums* $S_{p,p}(n)$*, where* $1 \leqslant p \leqslant r$*, are asymptotically boolean independent with respect to* $\phi(n)$ *as* $n \to \infty$*.*

Proof. This is a simple consequence of Theorem 7.2 and the fact that the family $(\zeta_{p,p})_{1 \leqslant p \leqslant r}$ is boolean independent with respect to the vacuum state Φ on $\mathcal{F}$. $\square$

Acknowledgment

I would like to thank the anonymous referee for very helpful remarks on the original version of the manuscript which have led to significantly clearer and more precise proofs of main theorems.

References

[1] Ph. Biane, Processes with free increments, Math. Z. 227 (1998) 143–174.
[2] K. Dykema, On certain free product factors via an extended matrix model, J. Funct. Anal. 112 (1993) 31–60.
[3] R. Lenczewski, Decompositions of the free additive convolution, J. Funct. Anal. 246 (2007) 330–365.
[4] R. Lenczewski, Operators related to subordination for free multiplicative convolutions, Indiana Univ. Math. J. 57 (2008) 1055–1103.
[5] R. Lenczewski, Matricially free random variables, J. Funct. Anal. 258 (2010) 4075–4121.
[6] N. Muraki, Monotonic independence, monotonic central limit theorem and monotonic law of small numbers, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 4 (2001) 39–58.
[7] D. Shlyakhtenko, Random Gaussian band matrices and freeness with amalgamation, Int. Math. Res. Not. IMRN 20 (1996) 1013–1025.
[8] D. Voiculescu, Symmetries of some reduced free product C^*-algebras, in: Operator Algebras and Their Connections with Topology and Ergodic Theory, in: Lecture Notes in Math., vol. 1132, Springer-Verlag, 1985, pp. 556–588.
[9] D. Voiculescu, Limit laws for random matrices and free products, Invent. Math. 104 (1991) 201–220.
[10] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory, I, Comm. Math. Phys. 155 (1993) 71–92.
[11] D. Voiculescu, Lectures on free probability theory, in: Lectures on Probability Theory and Statistics, Saint-Flour, 1998, in: Lecture Notes in Math., vol. 1738, Springer, Berlin, 2000, pp. 279–349.
[12] D. Voiculescu, K. Dykema, A. Nica, Free Random Variables, CRM Monogr. Ser., vol. 1, Amer. Math. Soc., Providence, 1992.
[13] E. Wigner, On the distribution of the roots of certain symmetric matrices, Ann. Math. 67 (1958) 325–327.